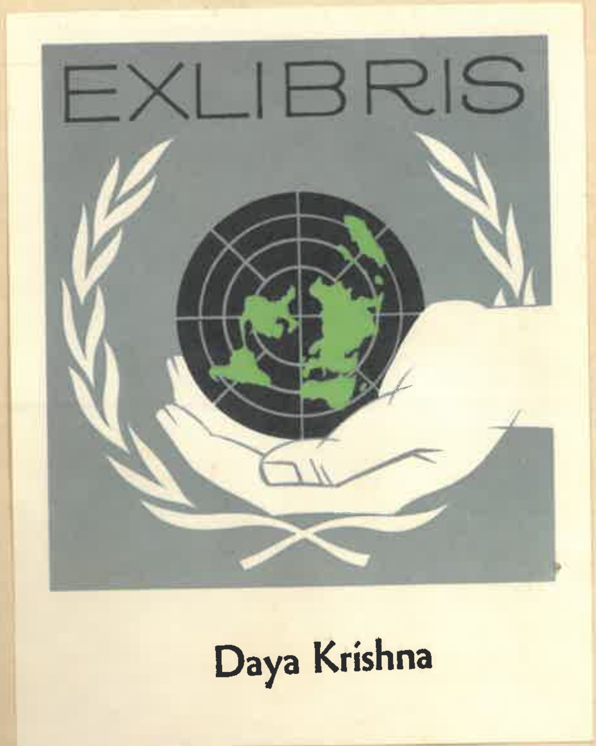

Daya Krishna

Modern Logic

Its Relevance to Philosophy

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PREFACE

PHILOSOPHY has always been technical, but perhaps never more so than in the recent past. Rarely has the technical spirit proved so arrogantly triumphant in the history of philosophy as in the recent impact of mathematical logic on a certain type of philosophical mind. True, there were worshippers at the altar of ordinary language who thought that, like the Delphic Oracle, it held the answer to all philosophical questions. But there was nothing like the mystery of symbols and the aura of rigour and proof to draw a charmed circle in which alone, it was thought, lay the key to the solution of all philosophical problems. In any case, the uninitiated had no right to question the pronouncements of those who knew and had found answers to questions that had troubled the thinkers for ages.

The story of the philosophers' fascination for logic is long. Yet, seldom have philosophical disputes been settled with the help of logic *alone*. Nor, for that matter, have any great philosophical discoveries been made through it. Aristotelian logic, or rather the logic associated with the name of Aristotle, had reigned supreme for more than two millennia and yet philosophers had continued to debate almost all the issues all the time, and differ. However, the claims of modern logic to the solution of ancient philosophical disputes were being made with far greater confidence than is usual in philosophical circles. Modern developments in logic seem to have finally shown the defects in many contentions made by philosophers before. Also, it was contended that no significant thinking in philosophy could be done without a deep and extensive acquaintance with modern logic. Modernisation of courses in the teaching of philosophy was everywhere supposed to start with an introduction of courses in modern logic which, it was suggested, should be made compulsory for anyone offering philosophy.

Side by side, developments in logic were becoming more and more autonomous, technical and independent of any specific concern with issues generally recognized to be philosophical. In fact, identifying logic with mathematics became the main issue around which the whole debate seemed to revolve and it was difficult to see how this could be of any great relevance to the solution of philosophical problems with which thinkers had been concerning themselves. If the two were really identical, then mathematics also would have the same relevance for philosophy as modern logic. Now, no one seems to have contended that a knowledge of mathematics is absolutely essential for philosophy or that significant philosophizing cannot be done without it. True, Plato is supposed to have prohibited the entry of non-mathematicians to his Academy; and true also that amongst philosophers of the first rank, there have been many who have made creative contributions to mathematics. However, the work of Plato himself can be understood without any extensive knowledge of mathematics. The great scholars of Plato's philosophy have scarcely been mathematicians or had even any extensive acquaintance with the subject. Similarly, the great creative thinkers in philosophy who have had no proficiency in mathematics far outnumber those who had some. Further, the philosophical and mathematical work of those who have been able to be creative in both falls apart and shows an independent intelligibility of the one from the other.

On the other hand, if logic is not identical with mathematics, then, in its modern form, it could not be more relevant to philosophy than it traditionally has been. The claim, therefore, that modern logic is relevant to philosophy in a *new* and far more radical way, needs examination.

The Seminar, whose papers have been published here, was organised in November 1967 by the Department of Philosophy, University of Rajasthan, with the help of a financial grant from the University Grants Commission. Mathematicians interested in philosophy and philosophers interested in mathematical logic were invited to contribute papers relating to that aspect of the issue they happened to be interested in. The Seminar lasted more than a week and was organised and structured by Dr. D. C. Mathur of the Department of Philosophy, University of Rajasthan. The papers ought to have been published long back, but due to some unforeseen difficulties it could not be done. However, they are offered now to a wider public interested in the issue. Some other works have appeared recently showing interest in the problems discussed in these papers.[1-5] It is our hope that these will further add to the awareness of the multiple facets involved in the issue. The papers deal with the problem at various levels of generality and specificity. There are those that deal directly with the subject and those that deal only indirectly with it and whose relevance to the subject has been discussed at length by the participants in the Seminar. But all of them add to our understanding of the issue in its essential diversity. Only the appendices are purely informative; they are included as background papers, specially for those who have no intimate knowledge of the subject.

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D.C. Mathur

RELEVANCE OF FORMAL LOGIC TO THE PHILOSOPHIC ENTERPRISE

THE last fifty years have witnessed unprecedented progress in formal logic. The trail blazed by Russell and Whitehead has been followed by a number of competent logicians and an attempt has been made to develop logistic systems which are completely formal, free from any empirical content. This has resulted in great technical virtuosity. Paradoxically, the craze for *clarity* and the cult of *unintelligibility* have joined hands to give the impression of massive profundity and esoteric wisdom! Not only the layman but even serious students of philosophy have been taken in. What could be more clear than starting with fewer and fewer primitive symbols, *formation* rules for defining a 'well-formed-formula' (WFF), a set of axioms which satisfy the criteria of consistency, completeness and independence, and a set of *transformation* rules, and then proceeding step by step to *deduce* theorems strictly in conformity with these rules? And yet, couched in an increasingly technical symbolism such formal systems recede farther and farther from the interests and concerns of honest, intelligent and bewildered men and women who come to philosophy for guidance and wisdom. Surely the time has come to assess the relevance of such formal logical systems to the enterprise of philosophizing.

I have deliberately used the word 'enterprise' because the activity of philosophizing cannot be neatly defined with the clarity and rigour of a logical system.

Here we find no axioms and no conclusions of deductive certainty. Here philosophic practice rather than philosophic theory is a better clue to its nature. History is replete with examples of first rate philosophical minds who have embarked on this adventure with a boldness, imagination and insight which leaves formal logic trailing behind. Right from Socrates, Plato and Aristotle to the moderns like Locke, Berkeley, Hume, Descartes, Leibnitz, Spinoza, Hegel, Bradley, Russell, Wittgenstein and Dewey we find that there are all kinds of arguments and discussions but none of them has anything like the rigour of a formal deduction. Mere clarity is a fake and is the best refuge of a person who has nothing significant to say. This is not meant to deny that, used with insight based on experience, clarity of expression can make a powerful combination. But by itself it is sterile.

Take Plato's Dialogues. What kind of logic is there? Is it deductive or inductive? We find no such logic in Plato's arguments. The charm and the perennial appeal of these Dialogues consist in their conversational style, a give-and-take sort of discussion—a kind of 'brain-storming' where participants use plenty of jest and humour combined with earnestness, where Socrates of the Dialogues asks novel and unsuspected questions and elicits answers and thus makes them 'see' a point from a fresh angle. The whole discussion has a generous sprinkling of myth and metaphor. Here we have a lot of arguments but no formal rigour; there is plenty of freedom and spontaneity and none of the constraint of a logistic system; there is an imaginative use of language but no watertight definitions—all aimed at attaining a fresh vision—a novel way of looking at things and thus gaining a fruitful insight into the problems of justice, beauty, love, power, goodness, happiness etc.

It is interesting to note that outstanding philosophers from the earliest times to our own day have been lured by the ideal of indubitable and absolutely certain

knowledge. Mathematics and its deductive certainty have served as a standard of all knowledge. Plato found such certainty in an intuitive knowledge of Ideas which were all supposed to be subsumed under the highest Idea of the Good. Yet it is strange to find that in actual philosophizing neither Plato nor others who were equally fascinated by mathematical ideas ever produced a single proposition as the conclusion of a formal argument. Paradoxically enough, this attraction for mathematical certainty has cut across party lines and has influenced the so-called idealists and realists, the rationalists and the empiricists, the absolutists and the pluralists alike. Even our own contemporaries—the logical atomists and the logical positivists—have fallen a prey to the allurement of formal certainty. But no philosophic advance—no philosophical discovery (and there have been significant discoveries in the history of philosophy) has ever been deduced from what was already known.

It is commonplace to mention that there is no logic (in its formal sense) of discovery either in science or art or in mathematics or philosophy. Those who have made significant advances have been persons who have had an acute sense of wonder and puzzlement at things which appear ordinary to ordinary people. They have sensed problems posed by the brute facts of *existence* and *experience* and have felt the need for rendering them intelligible. Or they have seen through the blind alleys in which some problems handed over by their predecessors have led into. They have tried to solve such problems not with the help of a formal discipline but by getting out of the ruts, turning the problem round as it were, having a fresh look and gaining a new vision till they 'saw' in an intuitive flash the solution to the jigsaw puzzle.

Let us consider a few examples. Descartes, Spinoza and Leibnitz were deeply impressed by the new mathematics of their day. Descartes and Leibnitz were outstanding mathematicians of their times. Descartes attempted to start from a clean slate and to found his system of knowledge on a self-evident and indubitable proposition. He hit upon the famous Cogito, Ergo Sum. But the way in which he established its self-evidence is hardly a matter of formal deduction. And his so-called proofs for the existence of external objects, other selves and for God have not the remotest resemblance to the rigour and necessity of a deductive system. It is idle to judge Descartes' contribution to philosophy in terms of these so-called proofs. His real contribution was to so define a thinking substance and a material substance as to make room for the freedom of the former and the comprehensibility of the latter in terms of mathematical laws of motion. The real heir to this insight was Newton and not Spinoza nor a host of others who attempted to define the elusive concept of substance. Again, Spinoza's brave attempt to develop a metaphysics on the basis of axioms and definitions was no more than a homage to the ideal of mathematical certainty. His real insight was in another direction. He conceived a vision of unification of the diverse and contingent facts which make up the fabric of nature. Such a vision of unification has always haunted not only philosophers but all thinking persons who contemplate the moving panorama of nature.

If we turn to the British empiricists, Locke, Berkeley and Hume, we find a fresh approach—a new way of philosophizing. Whereas the Continental rationalists tried, though unsuccessfully, to geometrize philosophy, the empiricists saw that if one wanted to know anything about the wide world one had to open one's eyes and alert one's ears—in short—to depend on sense-experience. The perennial appeal of Locke's *Essay*, despite its confusions and equivocations, lies specifically in reminding the arm-chair philosophers to open their windows and look around. Ironically enough, Locke and the sceptic Hume could not free themselves from the

grip of the mathematical ideal of knowledge. They too were in search of the rock-bottom indubitable certainties on which to build the edifice of knowledge. Locke's 'simple ideas' and Hume's 'impressions' were supposed to be the bricks, the atoms of sense-experience about which there could not be an iota of doubt. And yet if one reads through Locke's *Essay*, one cannot fail to notice the pathetic attempt to *derive* formally, as it were, the superstructure of all our knowledge. The empiricist Locke was enchanted by the rationalist ideal of necessary knowledge. One need not be told that Locke's *Essay* was a grand failure. Hume quickly realized this and through a stroke of genius separated rigidly the contingent empirical knowledge, on the one hand, and the necessary, non-empirical mathematical knowledge, on the other.

There is no need to multiply examples. I wish to mention that the powerful movements of logical atomism (pioneered by B. Russell and Wittgenstein of the *Tractatus*) and logical positivism of the members of the Vienna circle have directly drawn on the *insight* of Hume and yet have been strangely influenced by the ideal of *indubitable* knowledge. Russell with his knowledge of mathematics and his attempt to reduce the whole of mathematics to logic wanted to eat his cake and have it too. His philosophical 'guru' Hume had separated mathematical certainty from the contingency of empirical knowledge. This was Hume's *insight* at which he did not arrive by any formal reasoning. Russell and the early Wittgenstein tried to build up a metaphysical system of logical atomism which could be based on indubitable foundations and yet could account for all our empirical knowledge. This indubitable foundation was supposed to be the now famous *atomic facts* and the whole world was thought to be derived from a truth-functional combination of such basic or protocol sentences registering atomic facts. Such a reductionist programme failed and the search for an *ideal language* (interpreted extensionally or truth-functionally) which could picture or mirror the world of facts proved to be chimerical. It was never clear whether these ultimate constituents of the world were sense-data *a la* Russell or material points according to one interpretation of the *Tractatus*. The attempt to analyse nations, persons and physical objects into events or sense-data proved a ludicrous failure, and Russell—the formal logician *par excellence*—fumbled in calling them sometimes logical constructions, sometimes logical fictions or incomplete symbols. Russell's great interest in formal logic did not restrain him from committing ordinary errors in reasoning or a loose use of ordinary English and it is a well-known historical fact that G.E. Moore constantly performed this critical function of reminding him to use his words with precision.

It was soon realised that logical atomism was unfruitful and the reductionist programme could not be carried through. The logical positivists (Ayer, Carnap etc.) had another hunch (which was not arrived at by any formal procedure) and they attempted to cut the Gordian knot with the help of the famous verifiability theory of meaning. They divided all meaningful statements into (i) the empirically verifiable and (ii) the tautologous ones of mathematics and logic, and dismissed at one stroke all else as metaphysical non-sense. The attempt to formulate and reformulate the verifiability principle came across insuperable difficulties. Besides, this principle was neither empirically verifiable nor a mere tautology. It was soon realized that it was after all a metaphysical statement and hence meaningless by its own criterion. Here again it was the formidable Wittgenstein who, with a flash of insight (and not through deductive reasoning from premises), 'saw' that asking for *the* meaning of a word or statement was asking a wrong question. Wittgenstein of the 'Philosophical Investigations' realised that ordinary language was elusive and complex, and that the meaning

of a statement depended on the *context* in which it was used. Hence the proper question was not to ask *the* meaning of a statement but to ask for its *use* in certain contexts. Every statement had its own *logic* (in its wider sense). Wittgenstein abandoned logical atomism and dismissed the verifiability theory of meaning—for which Russell never forgave him. He conceived the task of philosophy to understand the diversity and complexity of the *uses* of statements and thus 'dissolve' philosophical puzzles by 'letting the fly out of the fly bottle.' Ryle, in his *Systematically Misleading Expressions*[1], Wisdom in his *Philosophical Perplexity*[2] and F. Waismann in his *Language Strata*[3] followed the new trail blazed by Wittgenstein of the mature period and ushered in a fresh approach.

This brief historical excursus has been undertaken to show that none of the significant advances in philosophical thinking was made by drawing conclusions from basic statements or protocol sentences. The so-called ordinary language philosophers have realized that language has a flexibility, an elusiveness, richness and diversity, because it *functions* to articulate equally diverse shades and nuances of meanings of our experiences. To reduce all meaning to cognitive meaning (for the sake of a supposed and dubious ideal of clarity) is to do violence to our emotional, aesthetic, moral and religious experiences. It is these experiences which make us distinctively human. Despite a host of logical positivists, men will continue to ask for the meaning of life (not of the word 'life') with its tragedies and comedies, hopes and aspirations, joys and sorrows.

Philosophy will not compete with science in giving us new knowledge of a cognitive kind and yet it will not

1. Proceedings of the Aristotelian Society, 1931-32.
2. Proceedings of the Aristotelian Society) Vol. XVI, 1936.
3. Published in *Logic and Language* edited by A.N. Flew (Second Series).

concede that cognitive knowledge is all that there is. Philosophy will not merely perform the critical task of a linguistic analysis to 'dissolve' pseudo-problems *but* it will remain open and responsive to the endless *possibilities* of human experience and extend the range of such experience through imaginative vision. It will not limit experience by an *a priori* fiat to mere sense-experience. It has a 'frontier' character—exploring and extending the limits of human experience. Formal logic and artificial calculi with their constraining influence are wholly irrelevant to the freedom and spontaneity involved in the philosophic enterprise. Every enterprise involves a risk but it is also true that those who take no risks can say nothing significant. To formalize philosophy is to trivialize it, mechanize it and to make it irrelevant to significantly human concerns. It may be noted in passing that even advances in logic have been made by a happy insight. There are no rules for it. The wind bloweth where it listeth. The discovery of three-valued Logic (true-false-possible) was made by Lukasiewicz in 1920 through an insight that the two-valued logic had to be abandoned to solve the problem of future contingents and escape from 'logical determinism' involved therein.

To conclude, it may be said that formal logic cannot clip the wings of philosophical imagination nor can it limit its freedom and spontaneity to give us new visions and open new possibilities of human experience. Great philosophers cannot be produced to order. The only quality required of them is authenticity of experience. I may go further and say that there is logic (not in its formal deductive sense) involved in all significantly human experiences—the moral, emotional, aesthetic and the religious—in the sense that they are more *meaningful* and *relevant* to us as human beings than fool-proof logistic systems satisfying all the criteria of consistency and completeness. It is time to realize that nothing is truly meaningful to us except that which

touches us emotionally, fires our imagination and stirs us to our very depths. An ineffable experience of deep love has a fullness of meaning for us which an abstract and uninterpreted logistic system may never have. For a man of logical clarity the experience and expression of a unique, all-possessing and overwhelming love may be a meaningless jargon and jumble of words which disregards the ordinary rules of grammar. But life is larger than formal logic and philosophy is that free enterprise which keeps close to life and attempts to articulate and render intelligible the infinite richness, variety and range of human experiences.

G.C. Pande

ON FORMALIZATION AND THE NATURE OF MATHEMATICS*

IT is almost a truism in cultural history that the marked emergence of formalization in any area or tradition is apt to adversely affect the exercise of creativity and growth in it. The elaboration of critical canons in art and literature, for example, has tended to encourage the production of stereotypes and decadence in more than one age and tradition. The attainment of maturity and classic form by creative activity, reflective and critical analysis leading to systematization of formal principles, increasing subtlety, complexity and refinement where major concerns and contexts tend to be submerged in minor, technical ones, decadence of creativity—this seems to be a fairly normal dialectical rhythm in the history of cultural enterprises. Philosophical traditions thus commence with a basic concern with human reality or values and after the creative expression of a *weltanschauung* attempt to formulate reasons for the values sought to be upheld for realization. The diversity and incoherence of points of view in any rich philosophical tradition leads them to a critical reflection over the criss-cross of debates and of arrays of alleged reasons and conclusions. This search for satisfactory reasons leads at a still more reflective and abstract level to the formulation of systematic criteria of what

* The author is grateful to Dr. Daya Krishna for valuable suggestions and criticism.

constitutes a proper reason. Logic thus emerges to explain the nature of rationality and provide abstract models of possible reasoning. After this, curiously enough, philosophy instead of entering the royal highway of coherence and assured progress seems to retire into the most contradictory and contentious schools busy in systematic elaboration and refinement and in minor disputes rather than major issues and doubts. The history of the later Schoolmen in Europe or of Navya-Nyaya in India constitutes an example of this process of increasing logical formalism by the side of a loss of creativity in the central concerns and issues of philosophical thought.

The reason for this situation seems to be that formalism tends to emphasize the *results* of reflective and critical enquiry as regulations for creativity. The logic or illogic of discovery, however, does not appear to be wholly identical with the critically formulated rules of achieved results. That is why progressive discoveries in any realm of experience and knowledge necessarily lead to critical re-formulations. Formalism tends to treat the forms apprehended by critical reflection as constituting a universal and final world while creativity depends on an uncommitted search or groping for the novel and the unknown. Thus reflection over linguistic usage produces its systematic description in the shape of grammar which is then apt to claim a prescriptive right over language and thus either be disregarded or hinder its growth. The role of Paninean formalism is an instance of this.

While individual or concrete reality is experienced directly, forms are apprehended indirectly through reflection over remembered experience and this largely means social experience explicitly recorded or preserved as an institution or artifact. The common sense of ordinary language, the tradition of scientific knowledge and the special tradition of formal enquiry themselves constitute a changing and contingent record of social experience which is presupposed in the analysis and systematization of forms. Thus, although formal concern is characteristic of logic and modern logic is a veritable apotheosis of formalism, even a science thus conceived to be purely formal is indirectly dependent upon the state and type of experience, knowledge and language from which its material is drawn. Logic has advanced, receded, stood still and advanced again. Within each logic, formalism portends a danger. Formalism in effect tends to deify a particular logic.

While the adoption of mathematical symbolism has given a new power of computation to extensional logic, the close logical analysis of mathematical proofs has led to greater elegance, systematization, re-formulation and even extension of the scope in mathematics. There seems to be, however, a manifest distinction between mathematics as a creatively growing and at least relatively autonomous kind of knowledge and the mere logical analysis of the consistency or demonstrative rigour of already discovered mathematical results. Connectibility with physical space or natural numbers frequently appears as the *terra firma* for belief in mathematical consistency while the validity of valuable branches of mathematics continues to be debated by some logicians.

The claim that mathematics is reducible to pure logic and that both are indistinguishable as a science of pure form has fateful implications and poses far-reaching philosophical questions. Belief in the perfection of mathematics has attracted the attention of philosophers through the ages in the West. Mathematics has been held as an example of ideal knowledge, at once universal and necessary and capable of illumining reality. Modern logic emphasizes the demonstrative character of mathematical knowledge which invests it with only a formal validity that can be converted into truth only if we choose to assert the primitive assumptions which underlie it. Unless the primitive assumptions were

absolutely primitive and undeniable, mathematics would simply be vast arrays of symbols capable of having any sort of meaning, true, false or indeterminate. The choice between these would then be dictated by the needs and results of practical applications. In that case mathematical truth would be pragmatic and empirical and hence tentative and uncertain.

If, on the other hand, mathematics can be demonstrated to follow from absolutely primitive and undeniable assumptions, it would restore the sovereign character of mathematical knowledge with its self-assured and universal sway. It would, however, also restore the ancient misgivings and questionings about the origins and justification of this authority of mathematics to legislate over nature. If mathematics is wholly an *a priori* science, if it is a science of pure forms and its symbols are uncontaminated by any empirical significance, how come that its formulations apply to nature? After all, in our zeal for the logical purity of mathematics, can we forget that the value, prestige and content of mathematics have throughout depended very closely on its practical applications and successes?

The question of the truth and hence of the nature of mathematics inevitably leads to metaphysical disputes of classical standing. Viewing mathematics as logic would either entail a Platonic 'realism' about forms and would relegate nature itself to the secondary status of a copy, or a Kantian Copernican revolution about knowledge which cannot avoid making knowledge a systematically tempered product instead of being the transparent revelation which it always claims to be. Or we might alternatively think of mathematics either as a tautologous self-cancelling device functioning catalytically in the process of knowledge, or merely as the systematic inventory of the general conditions of past successes in thought ascertained pragmatically. The first alternative would confer on mathematics the status of necessarily and essentially true knowledge, the second that of necessarily but phènomenally true knowledge. Both would respect mathematics as logic, a science of forms necessary and pure either because they express the ubiquitous essence of reality or because they express the inevitable nature of the mind. On the third view, mathematics would have a putative necessity and convenience rather than truth. On the fourth, no knowledge would ever be able to make the transition from probability to certainty.

Logical realism suffers from the difficulty of having to point out a characteristic and independent way of knowing these abstract reals or pure forms. Plato postulated a prenatal confrontation of the *nous* with the eternal realm of forms and later anamnesis. One might postulate a distinctive intuition of the ultimate pure forms. Unfortunately it is difficult to reconcile the notion of intuition with that of abstract, determinate and relational knowledge. Can relations or even qualities be apprehended except through abstracting comparison and contrast of the perceptions of individuals? If knowledge of forms is not direct but derived from experience it will lack necessity and universality. If it is direct it will be akin to perception and not mathematics.

Transcendental idealism, on the other hand, is unable to give a satisfactory account of 'categories.' Kant took them ready-made from Aristotle and Hegel's list neither wholly excludes the empirical nor is the deduction wholly rigorous. In any case, how is one to be certain that the so-called presupposition of knowledge or objectivity are not merely psychological habits with a biological, cultural and linguistic basis?

Hilbertian formalism seems to misconstrue the very nature of logic. Mere supposition and paper marks can not produce any operational necessity. If our assumptions are wholly arbitrary, there would be no difference between any logic of games and any mathematics. Besides, the notion of entailment is inseparable from an

indefinable but ineradicable feeling of logical necessity. The inconceivability of the contradictory has the same status. The logical mind is throughout controlled by a sense of objective necessity which, if not inherent at the start, cannot be derived later at all. If we have only suppositions to start with, we cannot reach necessity.

If that mathematics is true which we choose for its applicability, we cannot escape a pragmatic position on truth. If we accept the Humean sundering of factual perception and rational necessity, we would have to bid good-bye to the latter. If, in perceiving a fact, we apprehend no necessity, there is no way of apprehending any because there is no apprehension which is not derived from some perception or another. Either inference is only the explication of a wholly conventional meaning, an endless or circular transformation of sentences where the conclusion is not a knowledge at all, or it cannot avoid an inductive aspect. It has been objected that induction itself presupposes a general principle which cannot be derived inductively. It is true that general principles are not amenable to simple perception. Otherwise the tedium of thinking would be unnecessary. It seems then that general principles are the result of an economizing tendency in thought supported by linguistic habits. The knowledge of general principles is simply a working illusion, an *'avisamvadi-bhrama'*.

The problem of reconciling the necessity of mathematical knowledge with its applicability to the real world is only a special case of the inherent paradox of knowledge which is at once direct and indirect, a confrontation and transparent revelation of the real as well as an interpretation and construction of a world. It will not do to assume two sharply divided stages of knowledge or faculties of the mind because then it would be impossible to bring them into any relevant relationship. We can only assume that perception and reason, immediacy and reflection are twin aspects in an experiential continuum, which at the social level passes through the recurrent cycle of theoretic expectation and practical encounter. It is the attempt to close the gap between the two which leads to progress in knowledge. As a science, even though abstract and general, mathematics shares in this dialectic of creative and impredictable growth and is irreducible to any set of completely demonstrated formal rules. Mathematical forms are as inexhaustible as those in art, and the logic of either must remain incomplete.

Rajendra Prasad

MODERN LOGIC AND PHILOSOPHY: THEIR RELATIONSHIP

IN the past the relationship between logic and philosophy was taken for granted. Philosophers did not question the claim of logic to occupy an important part of the commodious mansion of philosophy, nor was any non-philosopher eager to offer it better accommodation. But in the last few years the situation has greatly changed. There are philosophers who are reluctant to call logic any more a branch of philosophy; they find it much more mathematical than philosophical, and therefore would feel happy if it is treated as a part of mathematics. This proposed transfer of logic from philosophy to mathematics would not, as I read the situation, be unwelcome to many mathematicians. Even some (so-called) neutral observers would regard all this only as a natural consequence of specialisation and growth, and may point to the fact that several disciplines, which once upon a time were branches of philosophy, are no more so considered. But, besides those who still consider logic an important part of philosophy either because the philosophical tradition so considers it, or because they think that it ought to be so considered on its own merit, there are some equally respectable philosophers who lean towards the other extreme and hold that logic is the *essence* of philosophy. For this latter view philosophy is nothing but logic. It is needless to say that neither the term 'philosophy' nor 'logic' has the same meaning in the usage of the exponents of these opposite views. In what follows I shall try to examine some important ways in which logic can possibly be claimed to be related to philosophy without making any attempt to define or describe in any detail either what logic is or what philosophy is, because, although an understanding of the meaning of these terms is necessary for understanding their relationship, my objective in this paper is to explore the nature of this *relationship* and not the nature of logic or of philosophy. I shall depend upon the philosophical common sense of the reader which, I think, is all that he needs to appreciate the present discussion.

In the present-day philosophical usage logic stands for a *formal* as well as *informal* enquiry. In the last fifty years it has made great strides in both directions and there is no indication that it will not continue doing so in the coming years. In fact, it is this dual development of logic which has upset the equilibrium and made people think afresh about the relationship between philosophy and logic. The development of formal logic in modern times as a highly technical enquiry, specially the attempts at axiomatization, has made it look much more akin to pure mathematics and much more different not only from an accredited branch of philosophy like metaphysics or epistemology but also from traditional logic as presented by its founder and developed by many later philosophers.

This development gives rise to the question 'Is logic really a part of philosophy or of mathematics?' But logic has also been developed into an informal but extremely powerful technique of analysis which can be used to dissect any philosophical (rather, any *conceptual*) problem. The use of this technique has been made so extensively in recent years that it has covered all the fields of philosophy, including even ethics, aesthetics and philosophy of religion. This leads to the question 'Is philosophy simply logic?' (or 'Is this kind of philosophy philosophy?').

I shall discuss first the relationship of philosophy to formal logic, and then to informal logic.

Logic is a *formal* enquiry in so far as it studies the *forms* of arguments. In this respect the traditional distinction between deductive and inductive logic is not very important because both are, in my opinion, concerned with argument-forms, and the differences between them are due not to their being different kinds of logic but to the different kinds of arguments which constitute their subject matter.[1] Moreover, for the purposes of this paper, the distinction between deductive and inductive logic can justifiably be ignored. Therefore, in this discussion I shall speak of logic as one enquiry.

Formal logic can be done non-rigorously or rigorously. It will be non-rigorous if it studies and catalogues the various argument-forms or rules of inference without organising them in an axiomatic system. It will be rigorous if it presents an axiomatic system. An axiomatic system has to be, by definition, rigorous. It is rigorous when *all* of its theorems are *logically* proved only on the basis of its axioms by one or more than one application of its rule (or rules) of inference. The construction of axiomatic or logistic systems is to a very great extent a twentieth century phenomenon, though in the history of Western logic attempts were made even in earlier times to axiomatize it. But I am not aware of any such attempt having been made in the history of Indian logic or of a system evolved by an Indian logician or mathematician, whether living or dead. Logic started as a non-rigorous enquiry and therefore examples of non-rigorous logic can be found even in the logical speculations of the Jainas and the Buddhists. The logic of Aristotle is also non-rigorous.

1. For a detailed discussion of the subject, see the author's 'The Formal Character of Logic' in *The Philosophical Quarterly*, April, 1961.

Logic, meaning by it formal logic, it seems to me, can be conceived as a *part* of philosophy, a *subject-matter* of philosophy, or a *tool* of philosophy. I shall explain the three relationships one by one.

I

(a) *Logic as a part of philosophy*

By 'part' here I mean the same thing as 'branch' or 'field.' As already pointed out, the philosophical tradition does treat logic as a part of philosophy and of no other discipline. Even now, so far as non-rigorous logic is concerned, people do not object to its being so treated. If any doubt is raised, it is about rigorous logic, i.e. logistic systems. Whether or not a logistic system be treated as a part of philosophy is very much a matter of definition and therefore the decision is bound to be, at least to that extent, arbitrary. But there does not seem to be any *prima facie* good reason for not calling it a part of philosophy. A logistic system represents the ideal of logic as a formal-scientific enquiry, and if it is suspected that, when made rigorous, logic gets mathematicised, it can be said that the various attempts to make logic rigorous have also very greatly contributed to logicising mathematics.

(b) *Logic as a subject-matter of philosophy*

Logic also is a subject-matter of philosophy in the sense that it gives rise to many philosophical questions. The attempt to deal with these questions leads to what may be called a philosophy of logic, just as a philosophical study of religion leads to a philosophy of religion. The discussion of the logical behaviour of religious utterances is not a *religious* but a *philosophical* discussion. In the same way, science also gives rise to a philosophy of science.

It may be asserted here that logic would, on the analogy given above, cease to be a part of philosophy, just as science, which is the subject-matter of the philo-

sophy of science, is not a part of philosophy. One may or may not accept this position depending upon how he wants to use the word 'philosophy'. However, according to the existing convention logic *can* be a part as well as a subject-matter of philosophy. It is a peculiarity of philosophy that it philosophises also about itself and we consider this philosophy of philosophy no less philosophical than, say, pure metaphysics. One may call philosophy of philosophy meta-philosophy. But that will not alter the situation in any significant way because meta-philosophy does not mark out an enquiry very much different from philosophy in so far as almost every question of philosophy can be, if it is not explicitly so, transformed into a meta-question.

Philosophy of logic is not a very recent growth. Bradley's *Principles of Logic* is full of philosophical explorations into logic. So is Dewey's *Logic* and Schiller's *Logic For Use*. But I have the feeling that the authors of these early works so fused together logical and philosophical issues that they not only very often lost sight of the distinction between them but also gave the impression to the reader, at least sometimes, that it does not exist. It seems obvious that the recognition of the distinction would contribute to clarity and precision. Rather, in any enquiry, not to see or recognize a distinction where it exists is a sign of intellectual immaturity if it is not just due to oversight. It is true, however, that the finer the distinction, the easier it is to fail to see it and what is perhaps important, to recognize it.

Logic has provided a large many problems and occasions for philosophical reflection and analysis. For example, the concept of validity is a logical notion and to define it, to lay down its rules, or even to organize them in a deductive system, are *bona fide* logical activities. But the question 'Is validity a descriptive or evaluative concept?' is not a question *in* logic but in philosophy. Similar questions can be raised about probability, possibility, necessity etc. If one argues that 'valid' is an evaluative term like 'good' and therefore cannot be defined by means of a purely descriptive expression without committing the naturalistic fallacy, then what he is doing is philosophy and not logic (= formal logic). Almost any concept of logic can be the analysandum of a philosophical analysis. Even ontological status of propositions has been responsible for some very intense and serious philosophical speculations. Logicians are fond of constructing models and artificial languages. These constructed languages raise some very extensive and basic philosophical issues. One may ask what philosophical purpose is fulfilled by a constructed language. Even its relationship with the natural language is worth investigating. We shall come back to this topic in the next section.

(c) *Logic as a tool of Philosophy*

In one sense logic is a tool of every enquiry in so far as every enquiry *uses* rules of logic in deriving its conclusions from the given premises. Rather, we shall not call anything an enquiry if it were not logical, i.e. if it did not conduct its explorations according to standard logical procedures. But logic is a tool of philosophy in a more specific sense. The techniques and results of logic have been used by philosophers in the formulation of their theories in a much more important manner than by non-philosophers. In the first quarter of this century, Russell used his logical researches in formulating his ontological theory of logical atomism. His logic provided him not only the rules of drawing philosophical inferences but also the tools by means of which he could formulate his ontology.

I think everybody will agree that conceptual analysis is one of the most important functions of philosophy. Logic has been used as a tool by several analytical philosophers in more than one way in the practice of philosophical analysis. Apart from the fact that logical

discoveries have aided and influenced philosophical thinking in several ways, there is one very important respect in which logic has very prominently played its role as a tool. I mean by this its role in the construction of artificial languages. Logical techniques and symbolism are very extensively used in constructing an artificial language. There are quite a few philosophers, who may be called reconstructionists, who hold that the construction of artificial languages is indispensable for philosophical analysis. Leaving open the question about the exact nature and objectives of philosophical analysis, we can find several motives, logical, ontological and analytical, responsible for the emphasis on constructed languages. (i) It has very often been said that the structure of an argument or proposition, when expressed in ordinary language, is not cent per cent clear. Thus, the chances of committing mistakes in assessing arguments in ordinary language are very great. It is therefore suggested that the construction of an artificial language, free from the defects of ordinary language, into which we may translate arguments, will obviate all such errors of evaluation. (ii) In recent as well as in earlier years many philosophers have maintained that ordinary language has misled some of their colleagues into constructing wrong philosophical theories. The remedy that is suggested is that an artificial language be constructed which will be in a sense an ideal language. This ideal language, according to some, will not only not mislead, but will also reveal the structure of reality, and thus ensure the construction of a sound ontology. The presupposition in this view is that the structure of the ideal language is the same as that of reality. (iii) Some philosophers hold that a constructed language can be treated as a model for unconstructed languages, whether technical or non-technical. Every language, whether artificial or natural, contains two sets of rules, namely, formation rules and transformation rules. The former determine which expression is well-formed and which is not. The latter determine the legitimacy of passing from one expression (or a series of expressions) to another. No language, if it is to serve its function as a means of communication, can permit either all possible combinations of expressions or all possible transitions from one to another. It is said that the formation and transformation rules of a non-constructed language are not completely explicit, whereas those of an artificial language certainly are.

The constructionist argues that the construction of a formal system enables the philosopher to understand better the logic of ordinary language. If OL stands for the ordinary language and CL for the model constructed language, then, he would say, by looking at the structure of CL we shall get, at least in some respects, a clear picture of the structure of OL because the formation and transformation rules of OL, not being clearly stated anywhere, its structure is not clearly visible, whereas, those of CL being precisely and unambiguously stated, its structure is self-revealing. One method of clarifying or analysing a concept is to find out and state the rules (of formation and transformation) which govern its use in the language-system to which it belongs. It is very easy to do this for a concept of any CL but extremely difficult for that of any OL. According to the reconstructionist scheme, the clarification of a concept of OL can be effected by providing a suitable analogue in CL. To clarify a concept is to *replace* it by one belonging to CL and giving this replacement will, as the claim goes, be equivalent to giving clarification because the concept which replaces the explicandum is all clear in so far as its place in the system is fully defined by the rules of the system.

I shall not comment upon this programme of philosophical clarification except for saying that it *might* succeed if the concept analysed is a technical one, that is, a concept belonging to a formal or empirical science. A constructed language *may* be a model for the technical

language of science, but I am very doubtful about its being so for any ordinary language. It may be that the constructed language is free from all the problems and puzzlements occasioned by ordinary language, but it is equally, if not more, important for the philosopher to solve these problems and to find out ways and means of exposing and avoiding these puzzlements. The reconstructionist claim when stated in an extreme form resembles that of the spirit-intoxicated metaphysician when he says that man has to face evils only so long as he is in this world; the moment he enters the other one, the world of spirit, which is completely untainted, he faces none because no evil exists there. All this may be true, but still it seems to me that, as long as we are in this world, it is extremely urgent to solve the problems posed by it.

II

(*d*) *Informal Logic: Logic as Philosophy*

It is very difficult to define informal logic. The best way to indicate what I mean by it, as it seems to me, is to give examples. It is this sense of 'logic' which is operative in such phrases as 'logic of morals,' 'logic of the word mind,' 'logical behaviour,' 'logical geography,' 'logical topography,' 'logical map' etc. In this sense the terms 'logical analysis,' 'conceptual analysis,' 'philosophical analysis' and even 'linguistic analysis' are synonymous. If philosophy is conceptual analysis—and I hold that it is, though I am not going to defend this view here—then the most important questions to be answered are: (1) what is the nature of this analysis?, and (2) what sorts of concepts are its analysanda?

One way to answer the first question is to say that philosophical analysis is logical analysis. Doing logical analysis is doing informal logic. I am afraid these notions cannot be explicated without involving some circularity. But circularity is not a sin if it helps to arouse the relevant intuitions necessary for effecting understanding. A concept can be analysed in so many ways which may or may not be logical. To find out its historical genesis in a child and trace its development in later life, or to explore its role in the social and cultural milieu, or even to examine its significance in the intellectual history of man, are respectable *scientific, factual* studies which may be undertaken by psychologists, sociologists, anthropologists and others. But they are not logical or philosophical studies. Logical analysis of a concept consists in presenting as accurately as possible an account of its behaviour vis-a-vis other related concepts, an account of the various possible ways of giving to it meaningful employment, of things which can and which cannot be said of it, of the things one is committed to say of it and of other related concepts as a consequence of saying a certain thing etc. Logical analysis, or informal logic, in this sense of the phrase, is identical with philosophy in so far as this method of doing philosophy can be applied to any concept whatsoever.

Coming to the second question (what sorts of concepts form the analysanda of logical analysis?), the reply is: any concept of general significance. In this respect non-scientific concepts are no less important than scientific concepts. There are several criteria for choosing a concept for analysis. It may be chosen because it is responsible for causing confusion and puzzlement, or because it occupies the central place in a certain system of concepts so that its analysis is likely to help the analysis or understanding of others, or because it has not been so far adequately or not at all analysed so that it arouses the curiosity of the analyst, or because it is used by so many different disciplines, or simply because it has gained a respectable place in the philosophical tradition, etc. Whatever may be said of the other functions attributed to philosophy, logical analysis of concepts seems to be distinctive of it as it has not yet been appropriated by any other discipline.

The present essay is an exercise in informal logic.

P.K. Sen

THE PROBLEM OF ENTAILMENT
A Case Study in the Relationship of Modern Logic to Philosophy

MODERN logic is relevant to philosophy, and there are two ways of showing this relevance. One may argue that it follows from the very nature of philosophical enquiry, or from the very nature of the logical, or from both that they involve one another. Again, one may try to show the relevance by a consideration of particular problems in logic or philosophy. I have preferred the latter in the present paper and therefore have taken the problem of entailment to illustrate the connection between philosophical and logical issues. It should be stated clearly, at the very outset, that the purpose is only to show the *connection* between them, not to give an adequate treatment of either. So the reader will be disappointed if he expects an adequate treatment of either the problem of entailment or of the philosophical problems which it involves.

The paper has been divided into three parts. In the first part I have undertaken to make clear the question, 'How is modern logic relevant to philosophy?' In the second part I have considered some theories of entailment in order to illustrate the way or ways in which logical and philosophical theories interact. In the third part I have called attention to two philosophical problems which are strictly continuous with the problem of entailment.

It is really very difficult to determine exactly how logic is related to philosophy. The difficulty springs mainly from two sources. The first is that the term 'logic,' even within the context of the recent developments of the subject, does not bear a single meaning; and the second is that philosophical and logical theories are *interdependent*—a fact which makes it hard to decide where a logical theory moulds, and where it is itself moulded by some philosophical theory. Since both these points tend to be overlooked I have decided to discuss them in some detail.

First, the ambiguity of the term 'logic'. The different senses of the term referred to above are clearly brought out by H.B. Curry. Curry's exposition of the point is almost impossible to improve upon. So I beg to be excused for quoting at length from his *Foundations of Mathematical Logic*:

> "The first sense is that intended when we say that 'logic is the analysis and criticism of thought.' (Johnson—*Logic*. Part 1, p. Xiii). We observe that we reason, in the sense that we draw conclusions from our data; that sometimes these conclusions are correct, sometimes not; and that sometimes these errors are explained by the fact that some of our data were mistaken, but not always; and gradually we become aware that reasonings conducted according to certain norms can be depended on if the data are correct. The study of these norms, or principles of valid reasoning, has always been regarded as a branch of philosophy. In order to distinguish logic in this sense from other senses introduced later, we shall call it *philosophical logic*.
>
> "In the study of philosophical logic it has been found fruitful to use mathematical methods, that is, to construct mathematical systems having some connection therewith.........the systems so created are naturally a proper subject for study in themselves, and it is customary to apply the term 'logic' to such a study. Logic in this sense is a branch of mathematics. To distinguish it from other senses, it will

> be called *mathematical logic*.
>
> "In both of its preceding senses 'logic' was used as a proper name. The word is also frequently used as a common noun, and this usage is a third sense of the word distinct from the first two. In this sense a logic is a system, or theory, such as one considers in mathematical or philosophical logic. Thus we may have classical logics, modal logics, matrix logics, Aristotelian logics, Kantian logics, etc."*

It is clear that the third of these senses is not important for our purpose. For, in considering the relevance of logic to philosophy we need not take into account this sense of the word 'logic' *in separation from* the first two. If logic exercises its influence on, and thereby becomes relevant to, philosophy then it does so only through the different systems which form its corpus. We may thus concentrate upon the first two senses. When we take the first two senses of the term 'logic', the question 'How is logic relevant to philosophy?' splits itself up into the following questions:

(1) How is philosophical logic relevant to philosophy?

(2) How is mathematical logic relevant to philosophy?

One may suppose that the first of these questions need not concern us because recent developments in the subject do not have anything to do with *philosophical* logic. The idea is current indeed that all progress in logic that we have made in recent years is in the field of *mathematical* logic; that philosophical logic is a thing of the past, that modern logic is the same as the mathematical and, further, it is only because we have left philosophical logic behind that we have been able to make such advance in logic as we have made. But this is a serious mistake. Philosophical logic is not a

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Pp 1-2.

thing of the past; modern logic is not the same as mathematical logic, and progress has been made in the former as well as in the latter. And, I want to add further, every genuine progress in mathematical logic is at the same time a progress in philosophical logic.

In order to see the point of these remarks we are to consider the relations in which mathematical logic stands to the philosophical. A system of mathematical logic, as pointed out by Curry, is a mathematical system *built with a view to solving problems of philosophical logic*, that is, problems about principles of valid reasoning. (One typical problem of philosophical logic is that of discovering the basic principles of valid reasoning; another is that of showing relationship between them.) This implies, among other things, that mathematical logic cannot be considered apart from what has been called philosophical logic. For this is a point which is not always realised—a mathematical system (a theory in abstract algebra, for example) is not *by itself* a logical system; it is a logical system, if at all, only in so far as it lends itself to the solution of the problems of philosophical logic. If a mathematical system does not have any bearing, direct or indirect, on the investigation of the principles of valid reasoning, then it will not be considered a logical system at all. This point is somewhat obscured by the fact that a mathematical system has sometimes a composite nature. That is, it may have a part which has and also a part which does not have any bearing on philosophical logic. Though only the former part of this system strictly deserves to be called 'logical', yet the system as a whole, including the latter part, may be, and usually is, called so because this latter part is a natural development of the former. But if we do not allow ourselves to be misled by this *extension* in the use of the term 'logic', then we shall see that mathematical logic cannot be considered in isolation from philosophical logic and, in a deeper sense, they together form a perfect unity. The point has also been

made by Curry:

> "......It would be a mistake to suppose that philosophical and mathematical logic are completely separate subjects. Actually, there is a unity between them.........Any sharp line between the two aspects would be arbitrary."[1]

If the relation between philosophical and mathematical logic is really this, there cannot be any progress in the latter which is not a progress in the former as well. As a matter of fact, we have made as much progress in the former as in the latter. Further, we have made some progress in *logic* only because we have not left philosophical logic behind, and that again because the term 'logic' applies, as our discussion should have made clear, to philosophical logic in the original and to the mathematical logic only in a derivative sense. Had we left philosophical logic behind we would have left logic itself behind. But, fortunately, we have not done this.

It is clear from the above considerations that we cannot ignore any of the two questions into which we have split up the question 'How is logic relevant to philosophy?' We are thus to consider the question 'How is philosophical logic relevant to philosophy?' as well as the question 'How is mathematical logic relevant to philosophy?' We shall begin with the first.

At first look, the question appears somewhat odd. Philosophical logic or investigation of the principles of valid reasoning has always been regarded as a branch of philosophy. So the question about the relevance of philosophical logic to philosophy seems to amount to: 'How is a branch of philosophy relevant to philosophy?' And the question, as it stands now, does not make much sense. What does the question then mean, if it means anything at all? I think the question does mean something and, so far as I see, it means 'How is philosophical

1. *Foundations of Mathematical Logic* Pp. 2-3.

logic, as a branch of philosophy, relevant to *other branches of philosophy?*'

I shall try to explain by consideration of some details how philosophical logic becomes relevant to other branches of philosophy. But before doing that I shall consider briefly the question about the relevance of mathematical logic to philosophy.

We have pointed out that philosophical logic is a branch of philosophy. So one may show the relevance of mathematical logic to philosophy by showing simply that mathematical logic is relevant to philosophical logic. And this is quite easily done. That mathematical logic is relevant to philosophical logic follows analytically, as we have explained, from the very idea of the subject. This is also the reason why, I think, the question 'How is mathematical logic relevant to philosophy?' is not taken to mean 'How is mathematical logic relevant to philosophical logic?' This is not so taken at least when it is felt to be a serious and difficult question. So the question that we have in mind when we ask 'How is mathematical logic relevant to philosophy?' is rather: How is mathematical logic relevant to philosophical problems which belong to branches of philosophy other than philosophical logic?

This question may be handled in two different ways. One may try to show that mathematical logic is relevant to philosophy by showing first that it is relevant to philosophical logic and then pointing out that the latter, in its turn, is relevant to problems outside the province of philosophical logic. Or, else, one may try to show the relevance in a more direct way by arguing that mathematical logic has a direct bearing on such philosophical issues.

We can see that the question we wish to consider here, viz., 'How is modern logic relevant to philosophy?' reduces itself into four questions at least:

(1) Does philosophical logic have any relevance to other branches of philosophy?

(2) Does mathematical logic have any relevance to philosophical logic?

(3) Does mathematical logic have any relevance to other branches of philosophy through its relevance to philosophical logic?

(4) Does mathematical logic have any direct bearing on other branches of philosophy?

It can be seen also that of these four questions the third is of secondary importance because an answer to it depends upon answers to the first and the second questions.

The second question again does not present us with any difficulty because it is quite clear how mathematical logic becomes relevant to the philosophical. It will be sufficient to consider the relation between conditional arguments on the one hand and a system of propositional calculus on the other. The more important questions are the first and the fourth. And of these two, I shall choose to consider the first, viz., the question about the relationship between philosophical logic and other branches of philosophy. The reason for this choice is partly that this question, I am afraid, tends to be ignored and partly because I do not feel myself competent to enter into the fourth question about the direct bearing of mathematical logic on philosophical questions which fall outside the scope of logic.

This is all that I wish to say about the difficulty of the question concerning the relation between modern logic and philosophy which follows from the ambiguity of the term 'logic.' It will be remembered that I said that the difficulty of the question is mainly due to two reasons. The other reason is that there is an interaction between philosophical and logical theories. A common assumption is that either logic will be determined by philosophy or philosophy by logic. But, actually, even in specific issues, philosophical and logical theories interact in the most complicated ways, making it extremely hard to determine the direction of influence.

The relevance of logic to philosophy I shall try to illustrate by a particular problem. It is the problem of entailment. I choose this rather than any other for a variety of reasons. The first is that this is one of the most fundamental problems in the entire domain of logic or philosophy. The second is that I hope to illustrate by this example how a problem of philosophical logic may be embedded in a mass of philosophical problems.

In the next part of the paper I shall thus consider some of the different theories of entailment with a view to catching them in the midst of philosophical theories.

II

The term 'entailment' was introduced by G.E. Moore in order to denote the relation which is the converse of the relation of deducibility. To say that p entails q is to say that q is deducible from (or follows from) p.

It has always been recognised that the relation of entailment is one of the necessary conditions of the validity of a deductive inference. In the opinion of the overwhelming majority of logicians this is also the sufficient condition. It is this fact which makes the concept of entailment so important for logic. If logic is an investigation of the principles of *valid* reasoning ('reasoning' here means 'deduction'), and if the relation of entailment, as obtaining between premise and conclusion, is a necessary condition for the validity of inference, then it is obvious that the concept of entailment occupies a central position among the concepts of logic. So it is also a basic problem of logic to give an adequate analysis of the concept. All the logical systems which have been developed so far are based upon some analysis or other of entailment. It is one of the greatest achievements of modern logicians that they have made this concept an object of explicit enquiry. I shall consider here some of the more important theories of entailment.

A. Let us take first the theory of Russell that entailment (which he calls 'implication') can be defined in terms of material implication.[2]

According to this theory, to say that p entails q is to say that it is not the case that p is true and q is false, or, what is the same thing, to assert $p \supset q$. But the theory is clearly false because it leads to at least two paradoxical consequences: (i) A false proposition entails any proposition whatsoever; and (ii) A true proposition is entailed by any proposition whatsoever.

The mistake which is committed in the theory is that when we say that p entails q we not only mean that it is not the case that p is true and q is false but we mean also that it is *impossible* that p is true and q is false; or, in other words, it is *necessary* that p materially implies q. But why does Russell commit this mistake which appears to be quite elementary? The reason is that he was trying to have a purely extensional logic which will not have anything to do with 'necessity'. But why was he trying to have an extensional logic? It is clear that he was trying to have this because it is only an extensional logic which is fully compatible with his basic empiricism. To reduce entailment to material implication is to eliminate necessity, and there is nothing which an empiricist will welcome more, for the concept of 'necessity', with its intimate connection with that of the 'a priori', has always been a headache to the empiricist.

* * *

B. Let us take next the theory of C.I. Lewis who identifies entailment ('implication' in his language also) with what he calls strict implication.[3] The merit of Lewis' theory is that he recognises the element of necessity in entailment which Russell sought to eliminate in

2. Russell and Whitehead—*Principia Mathematica*, Vol. I, p. 94 (2nd ed.)

3. C.I. Lewis—'The Calculus of Strict Implication,' *Mind*, 1914.

his theory. For, as Lewis explains, to say that p strictly implies q is to say that it is necessary that p materially implies q. But the defect in Lewis' theory shows itself in the further explanation which he gives of the notion of strict implication. To say that the material implication '$p \supset q$' is necessary is to say, according to Lewis, that its negation, i.e., '$p . \sim q$' contains a contradiction of the form $p . \sim p$. So the definition of strict implication in this theory comes to this: p entails q if and only if $p . \sim q$ contains a contradiction. And there are at least two defects in this theory. One of these two is more readily recognised than the other. We shall begin with this.

Like the paradoxes of material implication there are also paradoxes of strict implication, or, more precisely, paradoxical consequences which follow from the theory that entailment can be identified with strict implication. We take two of the more important paradoxes: (i) An impossible proposition (i.e., a proposition that contains a contradiction) entails any proposition whatsoever; and (ii) A necessary proposition is entailed by any proposition whatsoever.

We should note that neither of these two is a paradox when it is taken as a truth about strict implication. For strict implication is what Lewis defines it to be; and, according to the definition which Lewis gives, both the statement that an impossible proposition strictly implies every proposition and the statement that a necessary proposition is strictly implied by every proposition are necessary truths of logic. For when p is impossible it is a contradiction according to Lewis' idea of impossibility, and if p is a contradiction then, whatever q might be, $p . \sim q$ will contain the contradiction which p itself is. Again if q is necessary then the negation of q, i. e., $\sim q$ will be impossible; but, if $\sim q$ is impossible then it will be a contradiction and, consequently, whatever p might be, $p . \sim q$ will contain the contradiction which $\sim q$ itself is. So both when p is impossible and when q is necessary the conjunction $p . \sim q$ contains a contradiction, and, consequently, p strictly implies q. But necessarily true as they are of strict implication they are not only paradoxical but also definitely false of entailment.

Lewis was aware of these paradoxical consequences of the identification of strict implication with entailment. But he did not consider them to be paradoxes at all except in the sense of being strange and unfamiliar. He seeks, actually, to give a proof of each of them. I shall consider the proof which Lewis gives for the paradox 'An impossible proposition entails any proposition whatsoever.' The proof may be presented in the following manner:[4]

1.	$p . \sim p$	
2.	p	1, Simp.
3.	$\sim p . p$	1, Com.
4.	$\sim p$	3, Simp.
5.	$p \vee q$	2, Add.
6.	q	5, 4, D.S.

But the proof is invalid. Consider the final step in which Lewis deduces q from the combination of the two premises $p \vee q$ and $\sim p$. Why does Lewis believe that q follows from this combination of premises? Obviously because $\sim p$ is contradictory of p and, hence, if $\sim p$ is true then p cannot be true also; and if p is not true then the other component of the disjunction $p \vee q$, namely, q must be true. So in deducing his conclusion Lewis assumes that both the propositions p and $\sim p$ cannot be true together, or, in other words, the conjunction $p . \sim p$ can never be true. *But it is precisely this conjunction which has been taken as the primary premise of the deduction.* In an inference which is confessedly *per impossibile* we can start with an impossible proposition

4. This is a restatement of the proof given in Lewis and Langford—*Symbolic Logic*. Pp. 150-1.

as a premise but we cannot contradict this premise, either explicitly or implicitly, within the course of the same inference. In every inference there is a demand for consistency and Lewis' proof fails to satisfy this demand.

I should like to offer a few more comments on the nature of Lewis' proof. The fallacious steps, I am inclined to believe, are those in which he infers, from a contradictory conjunction, one of its conjuncts. I shall hazard here the remark that far from every proposition being deducible from a contradiction no proposition is deducible from a contradiction at all. To accept a contradictory proposition is to suspend, temporarily, the use of the law of contradiction. And when that is done no inference can, so far as I see, be drawn at all. An apparent exception to the rule that I lay down is a *reductio ad absurdum* argument. It would appear that in such an argument we start with a contradictory premise and proceed to deduce consequences from it. Now, it is true that the *reductio ad absurdum* argument presents us with difficulties. But I am inclined to believe that a *reductio ad absurdum* argument is only an equivalent transformation. And there is, I dare say, a fundamental difference between a step which is taken in accordance with a definitional equivalence and any other step. This difference, so far as I see, is that in a step which consists in equivalent transformation we only pass from one (linguistic or symbolic) *expression* to another, but in all other steps we pass from one *proposition* to another. It is only a step of the latter kind which I unhesitatingly take to be a step in inference. Russell once pointed out that inference is a species of assertion, that is, in an inference we make one assertion on the basis of another. But what is asserted is always a proposition. It does not strictly make any sense to say that we assert a *sentence* or any other form of linguistic expression. *We only use an expression in order to assert a proposition.* Thus when we pass from one expression to another we do not pass from one assertion to another.

The strongest argument against taking an equivalent transformation to be an inferential step is that an inferential step is always taken in accordance with a *logical principle* while an equivalent transformation is always made in accordance with what may broadly be called a *linguistic convention* (or rule). And the difference between a logical principle and a linguistic convention does appear to be quite irreducible.

But there are also difficulties in the view that a *reductio ad absurdum* is never a genuine inference. There is at least one consequence of this view which I am not at all inclined to accept. This consequence is that an argument in which we pass from one analytic proposition to another cannot also, on this view, be regarded as a genuine inference. Let me explain the point briefly. One necessary truth about entailment is that if p entails q then ~q entails ~p; from which it follows that if ~q does not entail ~p, p does not entail q either. Now, when both p and q are analytic propositions, both ~q and ~p are contradictions and, consequently, ~q cannot entail ~p according to the view that nothing can be deduced from a contradiction. But if ~q does not entail ~p then, as we have already recorded, p does not entail q either. It is mainly for this reason that I hesitate to conclude that it is impossible to deduce anything from a contradiction or that a *reductio ad absurdum* argument can never be a genuine inference.

But we cannot enter into all the problems that arise at this point. This is rather a digression. I have nevertheless made it in order to indicate the wide range of problems—all genuinely philosophical—involved here. A few of them are: (1) the nature of the process called inference, (2) the relation between a sentence and what is called, by philosophers, a proposition, and (3) the difference between a linguistic convention and a logical principle. I am in any case convinced that Lewis' proof of the thesis that an impossible proposition entails any proposition whatsoever is invalid. And the reason, as I have said, is that the proof suffers from a basic inconsistency.

We have said, it will be remembered, that there are at least two defects in Lewis' theory. One of these is that the theory leads to some paradoxical consequences. This defect, which we have discussed at some length, is readily recognised. But the other defect, which is philosophically even more significant, is not so readily recognised; on the contrary, in some quarters, this is regarded as a definite merit of the theory. This we shall discuss now.

According to Lewis' definition of entailment, to say that p entails q is to say that it is necessary that $p \supset q$. But to say *this* is to say that the contradictory of this material implication, i.e. $p.\sim q$, is impossible. But what again is the meaning of saying that this conjuction is impossible? This means, Lewis explains, that the conjunction contains a contradiction.

Now it is beyond doubt that if p entails q then '$p \supset q$' is necessary and '$p.\sim q$' is impossible, but what is not beyond doubt is that if $p.\sim q$ is impossible then it necessarily contains a contradiction. That is, it is not beyond doubt that if '$p \supset q$' is necessary then it is necessarily analytic—for Lewis' theory obviously amounts to this because if the contradictory of a proposition is contradictory then the proposition itself is analytic. Lewis is simply assuming here the analytic theory of necessity. But the analytic theory fails, in this form, in the case of entailment even if it does not in any other case. I shall discuss this point in the next part of the paper in greater detail. I want to ask now the question why this view has been taken at all by Lewis. One reason is surely that this gives a very simple account of entailment. But another, and the deeper, reason is that Lewis was talking from an empiricist point of view. Russell, from the same point of view, eliminated altogether the element of necessity in entailment. Lewis wants to ensure that if

entailment is a necessary relation then it is also a relation which is analytic.[5]

One can see the importance of the host of philosophical problems which are involved in Lewis' theory of entailment for purely logical (both philosophical and mathematical) issues when one considers the fact that Lewis' definition of entailment only gives a theoretical background to the technique called 'the method of truth table' and other related techniques of mathematical logic. When we assume in truth-functional logic, for example, that the inference '$p \therefore q$' is valid if and only if the material implication '$p \supset q$' is a tautology, we just take this theory of entailment for granted. And that is precisely the reason why the method of truth table leads to the same paradoxical consequences. Consider the inference:

It is raining and it is not raining.
$\therefore$ $2+2=4$.

Should we consider this inference to be valid? Surely not. But apply the method of truth table. You will find that the argument turns out valid according to that method. The reason is that the material implication which we form by taking the premise of this inference as antecedent and its conclusion as consequent is really a tautology. This is due to the fact that the antecedent of this material implication, being a contradiction, is false under all possible circumstances and, consequently, the material implication itself is always true. Take again the inference:

Russell is a philosopher.
$\therefore$ Either the Earth is round or the Earth is not round.

Should we consider this inference to be valid? Surely

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5. For a more detailed treatment see my paper 'The Analytic View of Entailment' in *The Journal of the Indian Academy of Philosophy*, Vol. II, 1963, Calcutta.

not. But it shows itself to be valid in the method of truth table; for the material implication corresponding to this inference is again a tautology because it is a material implication with a consequent which is true under all possible circumstances.

* * *

C. I shall discuss one more theory of entailment. This theory was also suggested by Russell and has been accepted, in one form or another, by many logicians, including Tarski and Lukasiewicz.[6] The theory behind the method of 'refutation by logical analogy', as expounded by I.M. Copi, is also essentially the same as this theory. We shall formulate the theory here by following the line suggested by Russell.

In formulating this theory we may take our start from the point which was also our point of departure in Lewis' theory of strict implication. When A entails B (or B is deducible from A) the material implication '$A \supset B$' is necessary. But what exactly is the meaning of saying that this material implication is necessary. A material implication, being a (compound) proposition, cannot be 'necessary' in the strict sense according to Russell. For necessity, possibility or impossibility cannot characterise a *proposition*. It can characterise only what he calls a propositional function.[7] There is, Russell thinks, a parallelism between the concept of existence on the one hand and these modal concepts on the other. [Existence also belongs only to propositional functions and not to individuals as many have usually supposed.] So when we say that '$A \supset B$' is necessary what we mean is only that it is a substitution instance of a propositional

6. Tarski—'On the Concept of Logical Consequence' in *Logic, Semantics and Metamathematics;* Lukasiewicz—*Aristotle's Syllogistic*, Section 5; also Reichenbach— *Elements of Symbolic Logic*, Section 23.

7. Russell—'The Philosophy of Logical Atomism' in *Logic and Knowledge*, p. 231; *Introduction to Mathematical Philosophy*, p. 156.

function '$p \supset q$' which is necessary. But to say that the propositional function '$p \supset q$' is necessary is to say only that every proposition (material implication) which is a substitution instance of this propositional function is true. Again, if a propositional function is of this nature then, obviously, the universal quantification of it is true. From the above it follows that A entails B if and only if the universal quantification of the propositional function of which '$A \supset B$' is a substitution instance is true. Now, the universal quantification of an implicative (propositional) function has been called a *formal implication* by Russell. If, further, we adopt the convention of calling a substitution instance of a propositional function a *specification* in relation to the universal quantification of the same propositional function, we can say, according to this theory of entailment, that A entails B if and only if '$A \supset B$' is specification of some formal implication which is true. We may illustrate the point of this theory by the following example:

The inference

"It is raining.

∴ Either it is raining or it is snowing."

is valid. But to say that it is valid is to say that 'It is raining' entails 'Either it is raining or it is snowing.' But to say this is equivalent to saying that the material implication 'It is raining ⊃ (Either it is raining or it is snowing), is necessary. But this means only that the propositional function '$p \supset$ (Either p or q)' is necessary, that is, such that every material implication which is a substitution instance of this function is true; or, in other words, the universal quantification of this function, which is the formal implication,

'(p) (q) ($p \supset$ (Either p or q))'

is true.

This theory of entailment appears to have been given the status of the Official Theory by the practising logicians. But, unfortunately, this theory involves the same kind of paradoxes that we encounter in the theory of strict implication. I only bring out two of the more important paradoxes:[8]

(i) A proposition which is a substitution instance of an impossible formula entails any proposition whatsoever. (When we say that a formula is impossible we mean that every proposition which is its substitution instance is *false*.)

(ii) A proposition which is a substitution instance of a necessary formula is entailed by any proposition whatsoever.

I shall explain now how we arrive at the first paradox.

Let A stand for any proposition which is a substitution instance of an impossible formula S, and B for any proposition whatsoever which is a substitution instance of some formula or other T. $A \supset B$ is, in that case, a substitution instance of the complex formula $S \supset T$. But if S is impossible then $S \supset T$ is necessary. This follows from the fact that for every substitution instance of S the corresponding substitution instance of $S \supset T$ is true because it is a material implication with a false antecedent, and a material implication with a false antecedent is always true. But if $A \supset B$ is thus a substitution instance of a necessary formula then A entails B. Since A stands for any proposition which is a substitution instance of an impossible formula and B for any proposition whatsoever this amounts to a proof that any proposition which is a substitution instance of an impossible formula entails any proposition whatsoever.

8. While working on the nature of entailment I came to discover these paradoxes in 1962 and incorporated this discovery in my First Annual Report as a Premchand Roychand Student of Calcutta University. I was not a little surprised to find that they had not been detected earlier. For amore detailed treatmeat of these paradoxes see my article 'The Paradoxes of Formal Implication' in *The Journal of the Indian Academy of Philosophy*, Vol. IV, 1965.

We can show, by following a similar line of argument, that a proposition which is a substitution instance of a necessary formula is entailed by any proposition whatsoever. In this argument we shall draw mainly upon the logical facts that every substitution instance of a necessary formula is true and that a material implication with a true consequent is always true.

To feel the full impact of these poradoxes one is only required to consider the following inferences:

(1) It is raining and it is not raining.

$\therefore 2+2=4$.

(2) Russell is a philosopher.

∴Either the Earth is round or the Earth is not round.

Each of these inferences is valid according to the definition of entailment we are considering now. For the premise of the first is a substitution instance of an impossible formula, and the conclusion of the second is a substitution instance of a necessary formula.

It will be remembered that we derived earlier the same consequence about precisely the same inferences from the theory of entailment which identifies entailment with strict implication. And this is exactly what is to be expected because there is a strict correlation between the two theories. Their correlation and the immense importance of the points that we have raised may be realised by a consideration of the following fact.

There are two *distinct*, but often confused, uses of the technique called the method of truth table. The first consists in taking the material implication corresponding to the inference which is to be tested for validity, and then deciding through a truth table whether this material implication is a tautology or not. The second consists in taking the implicative formula of which this material implication is a substitution instance, and then deciding in a truth table whether or not this formula is necessary. The definition of entailment presupposed in the first use of the method is the one which accords with Lewis' theory of strict implication, and the definition presupposed in the second use is the one which accords with Russell's theory of formal implication. The paradoxes show in each case that the method of truth table is not an entirely correct method of testing the validity of an inference.

Consider now the question why Russell was led once again to a mistaken view of the nature of entailment. The reason, in the case of formal implication, is the same as in the case of material implication. Russell, as a faithful empiricist, was trying again to analyse away the element of necessity which characterises every deductive inference and, consequently, the relation of entailment. Thus when he calls a formula necessary he does it only by courtesy. For what he means is only that every substitution instance of the formula is true (as a matter of fact). We need not say, Russell seeks to show, that the conclusion which is entailed by a premise follows *necessarily* from the premise. It is sufficient to say that the inference belongs to a form such that no inference belonging to it *does actually* combine a true premise with a false conclusion.[9] But this attempt at eliminating necessity from deduction fails utterly.[10] The implicative formula of which the material implication behind a valid deductive inference is a substitution instance is fundamentally different in kind from an ordinary implicative formula. Compare the two formulas :

(I) x is a man $\supset$ x is mortal.

(II) $p \supset (p \vee q)$

Each of them is such that no substitution instance

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9. It is obvious that Russell is trying to do here the same thing with logical necessity as Hume tried to do with the causal one.

10. See Pap—*Semantics and Necessary Truth*, p. 368.

of it is false. But in the case of (II) we can say that it is *impossible*. that a substitution instance of it should be false. In the case of (I) we cannot simply say that. It is possible that an individual should satisfy the antecedent but not the consequent of this implicative formula.

* * *

I now bring to a close my discussion of the different theories of entailment through which I was trying to show several points of contact between logical theories and philosophical issues. In the next part of my paper I shall focus attention on two philosophical issues which, I believe, are strictly continuous with the problem of entailment.

III

A. The first problem is that of the possibility of *synthetic a priori* propositions. The relation between this and the problem of entailment is that a careful investigation of the nature of entailment shows that there surely are senses in which *synthetic a priori* propositions exist.

We may begin with the very statements of entailment itself. A statement of entailment of the form 'p entails q' is necessary and, consequently, *a priori*. There is at least one sense of 'analytic' in which a proposition of this from is not necessarily analytic. Let us take the sense in which Kant used the term. In his use a proposition is analytic if and only if it can be validated by the law of contradiction *alone*. But all entailment statements cannot really be validated by the law of contradiction taken by itself. If it were so, then the validity of every deductive inference could be established by reference to this law alone. But this we know to be impossible. In fact, even such a simple entailment statement as '(It is raining)' entails '(Either it is raining or it is snowing)' cannot be validated by this law taken by itself.

In order to avoid such difficulties philosophers who uphold the analytic theory of the *a priori* have given a different definition of 'analytic'. According to this definition, a proposition is analytic if and only if it is validated by *some* logical principle or other. It is true that when the definition is changed in this way it becomes difficult to maintain that an entailment statement is ever non-analytic. For if entailment is the relation which, when obtaining between premise and conclusion, makes the inference valid, and if an inference is validated by logical principles alone, then it follows that an entailment statement (i.e., the statement saying that the premise entails the conclusion) will be validated by logical principles alone, i.e., by the same logical principles which validate the inference. But the difficulty which is generated by this solution is that the logical principles by reference to which the term 'analytic' is now defined become, according to it, *synthetic a priori* truths.

In order to avoid this new difficulty philosophers have suggested a further revision of the definition: A proposition is analytic if and only if it is either a principle of logic or something which can be validated by some principle of logic or other. This definition ensures that no *synthetic a priori* truth can arise in the field of logic.

But Moore, who introduced the concept of entailment, did not think that logic is the only field in which we may encounter entailment. One example of this non-logical entailment is the relation that obtains between a proposition of the form 'x is red' and the corresponding proposition of the form 'x is coloured.' If it is a genuine example then the corresponding entailment statement is synthetic as well as *a priori*. But I am not going to enter into the question of non-logical entailment. For that falls outside the scope of my paper. I have only mentioned it because there may be a basic affinity between logical and non-logical entailment. And, if there is,

an investigation into the nature of the first might necessitate that of the second also—showing once again the unity and continuity of logic and philosophy.

* * *

B. I shall conclude my paper with a brief reference to the problem of external and internal relations. It will be remembered that it is in connection with this problem that the concept of entailment was introduced by G.E. Moore. I think that the problem of internal and external relations is a genuine philosophical problem and is not metaphysical in the sense of being non-significant.

According to the theory of internal relations, every relation is necessary in the sense that if a certain individual x stands in the relation R to another individual y then it is impossible that x should exist apart from this relation to y. Or, in other words, if an individual x stands in the relation R to an individual y and an individual z does not stand in the same relation R to y then, *necessarily*, x is not identical with z. Moore points out, against this theory, that 'x is related to y by way of R and z is *not* related to y by way of R' does materially imply but *does not entail* 'x is not identical with z.'

Since the time of Moore the concepts and techniques of modern logic have been applied to the problem of internal and external relations. For the problem reduces to the question: Are all relational statements necessary? And this question whether statements of a particular type are necessary or not can be assimilated without difficulty within purely logical questions. *Prima facie*, no relational statement can ever be necessary, because all of them are singular. A typical relational statement is 'Socrates is the teacher of Plato.' And such singular statements are inevitably contingent. But one may argue that all relational statements are not of this kind; that is, there may be relational statements which are not strictly singular. If the individuals that are related enter into the relational situation as instances of properties, that is, as *defined* by them, then the statement of this relation would be 'potentially' universal (an idea of Bradley's) even though it might appear to be singular. That is, a relation R might obtain between the individuals x and y simply by virtue of the fact that x is an instance of the property P. In that case the question whether the relation R between x and y is necessary reduces itself to the question whether being an instance of the property P *entails* being related to y. And now we are back again to the problem of entailment. The empiricists have naturally tried to maintain that being an instance of P entails being related to y only if they are analytically related. The opponent may ask to consider, at this point, such relational statements as 'This flower, which is blue, *belongs* to the class of red thing.' But we cannot enter now into all the controversial issues which this question will raise. I am satisfied if I have succeeded in showing the connection between the problem of entailment and the problem of internal and external relations.

L.C. Mulatti

LOGIC AND PHILOSOPHY

IN this paper I shall talk about the role or function of logic in philosophy or the relationship between the two.

Let us begin by noting what logic means. One of the usual definitions of logic is that it is the science of valid reasoning. David Mitchell says :

> "Elementary logic is the study of the forms of valid arguments, and more widely, of the different types of proposition, which are logically true."[1]

The business of logic is to consider the various ways in which we actually think and to determine what makes these ways valid or invalid as against true or false. Since our thinking consists of propositions, the logician is concerned also with the character of these propositions and the various relations between them. He is not worried whether these propositions are actually true or false. His concern is with their different forms and their interconnections. Thus, the logician's enquiries are completely general and formal.

This is true, whether the logic we consider is traditional or modern, for there is no essential distinction between the two, though in appearance the two look vastly different. In fact, the difference is so great that

1. *Introduction to Logic.* London : Hutchinson University Library, 1962, p. 9.

if a logician not acquainted with modern researches in logic, tries to read a treatise on modern logic, he feels baffled by both the form and the content of the treatise. This novelty is due to nothing more than the fact that modern logic has devised its own notation or symbolism and speaks in terms of a highly technical jargon. The novelty is also due to the fact that modern logic has become more general and formalized than traditional Aristotelian logic, thus including in its sweep the whole of mathematics and much more, while traditional logic was merely syllogistic. But all this is a difference in degree rather than in kind. For, in essence, modern logic is a continuation and culmination of traditional logic.

While it is relatively easy for us to give a precise account of logic, it is very difficult to do the same in the case of philosophy. For, it is a common saying that no two philosophers ever agree as to their subject matter. Though this saying is an exaggeration, it none the less indicates that it is extremely difficult to give an agreed account of what constitutes philosophy. But there have been always in the history of philosophy broad trends which should enable us to give a rough and ready notion of philosophy. Thus, one of the most typical definitions of philosophy that has found favour with a vast majority of philosophers down the ages is that it is a study of the nature of ultimate reality. Russell, though he himself cannot be regarded as a typical representative of this kind of philosophy, says :

> "It is the attempt to conceive the world as a whole by means of thought."[2]

The activity of the philosopher, according to this conception, consists in constructing massive visions and imaginative pictures of reality as a whole and providing them with some kind of rationalisation. Embedded in such constructions is the persistent antithesis between

2. *Mysticism and Logic*. Penguin Books, p. 9.

the discrete things of experience and what is regarded as the ultimately real. This antithesis carries the implication that the particular things of experience are somehow unreal, or a mere appearance, while the ultimately real is either concealed behind them or is altogether beyond them. There was thus, if I may say so, an obsession with other-worldliness characterising most traditional philosophers. The traditional philosopher looked upon the activities of other mortals with contempt and claimed for himself a superior status, since, according to him, they were entangled in illusions he had successfully transcended and thus was in possession of the truth. This is the philosophy in the grand tradition of Sankara and Kapila, Spinoza and Kant, Hegel and Bradley. Such philosophy has generally been called, though with contempt, as metaphysics by modern philosophers and for the sake of convenience we shall also call it the same.

How then is philosophy in this sense—in the sense of metaphysics—related to logic? Or, to shift the emphasis, how is logic related to it? The answer is that logic has no special relationship, no special contribution to make to it. It is related to metaphysics in the same way in which it is related to any other branch of knowledge—history, literature, sociology or physics; that is it supplies the principles according to which the thinking in these branches is conducted. (Hence the hackneyed saying that logic is the science of all sciences.) One may object to this way of expressing the relationship between logic and metaphysics by saying that it is not so thin. It may be urged that metaphysics often derives both its character and content from logic. Did not the structure of Leibnitz's metaphysics, for example, spring from his logical doctrine? Particularly, his conception of the monad as a substance, which contains all its states within itself and whose history consists merely in a gradual unfoldment of these states, is derived, it is claimed, from his logical theory that all propositions have one and the same logical form which consists in assigning a predicate to a subject—a theory which he shared with all traditional logicians, including Aristotle.

> "Every proposition is ultimately reducible to one which attributes a predicate to a subject......Every predicate, necessary or contingent,......is comprised in the notion of the subject. From this proposition it follows, says Leibnitz, that every soul is a world apart; for every soul as a subject has eternally as predicates all the states which time will bring it."[3]

Or, if one wants a more convincing example one may consider the case of logical atomism which directly involves modern logic itself unlike Leibnitz's metaphysics. Logical atomism is certainly metaphysics, though not in the grand old style. Just because its authors are such unorthodox people as Russell and Wittgenstein, we cannot say that it is not metaphysics. On the contrary, as Urmson says :

> "Logical atomism was presented as a superior metaphysics which was to replace inferior ones, not as an attack on metaphysics as such. Indeed... logical atomism is one of the most thorough-going metaphysical systems yet elaborated. This is true in spite of the antimetaphysical strain...in Wittgenstein's *Tractatus Logico-Philosophicus* alongside of the metaphysics. For breadth of sweep, clarity, detailed working out and consistency, it can have few rivals."[4]

Logical atomism, as is well known, pictures reality as consisting of an infinite number of atomic facts which are mutually independent and which, in turn, consist of objects related in a particular way. These objects, like the facts to which they give rise, are also mutually

3. Bertrand Russell, *A Critical Exposition of the Philosophy of Leibnitz*. London : Allen & Unwin, New Edition 1937. Pp. 9-11.

4. J.O. Urmson, *Philosophical Analysis*. Oxford, Clarendon Press, 1956, Pp. 4-5.

independent. They are simple. Facts, then, are complex only in the sense of being constituted by objects and not in the sense of containing further facts; for, all facts are of the same kind, that is, they are all atomic facts which do not have other facts as their parts.[5] (Let us ignore here the extremely important complication arising out of the recognition of general facts, negative facts, and facts corresponding to intentional functions; for, a consistent logical atomist would forbid the recognition of these. The atomic fact is the paradigm of facts.) Though the facts are configurations of objects, the objects are not regarded as ultimate constituents, because they cannot have their existence outside of some fact or other, and therefore, are by themselves abstractions like Aristotle's first substance. They are only logically and not ontologically prior to facts. When we are able to identify or characterise an object, we have already a fact of which it is constituent. Objects, thus, represent the logical limits of the analysis of our knowledge. It is for this reason that logical atomists, especially Wittgenstein, were unwilling to give concrete examples of facts.

Objects or things are divided by Russell into particulars and universals or constituents and components. Constituents are individually identifiable things, while the components are either characteristics or relations. Every fact must contain, according to him, at least one constituent and one component.

This, in outline, is the metaphysics of logical atomism. Now, such a metaphysics, even according to its founders, was derived from a certain view of logic.[6] Both Russell and Wittgenstein were primarily interested in the problems of the foundations of mathematics and logic

5. Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*. Translated by D.F. Pears & B.F. McGuinness, London. Routledge & Kegan Paul, 1961. See especially Paras 1,1.1, 2, 2.01, & 2.0272. See also B. Russell's article. "The Philosophy of Logical Atomism" included in his *Logic & Knowledge*, London, Allen & Unwin, 1956.

6. B. Russell, *Logic and Knowledge*. p. 178.

and their metaphysics was a result of their investigations into these. Russell's task was to devise a logic from which the whole of mathematics could be deduced. He thought he succeeded in devising such a logic which he presented in his *Principia Mathematica,* in collaboration with Whitehead, and which is called truth-functional or extensional logic. This logic, roughly speaking, starts with an extremely small number of propositional variables based on a few basic and purely logical notions, and derives from these, together with some logical operators, all the other propositions of both logic and mathematics, including the most complicated ones. The propositions thus derived are called complex. Since their truth or falsity completely depends on the truth or falsity of the simple propositions they are also called truth-functions of simple propositions.

Russell's apparent success in deducing mathematics from logic led him to believe that since mathematics is applicable to reality, the structure of his new logic must correspond to the structure of reality. (The possibility of alternative calculi was not clearly realised then.) For, how else could he succeed in his deduction? He argued, therefore, that his logic must be a skeleton or a syntax (minus the extra-logical vocabulary) of any natural language. A natural language is by itself inadequate and imperfect, and the logic as set out in his *Principia Mathematica* brings out its accurate syntax.

> "The world is thus...of identical structure with, and to be perfectly representable by, a language with the structure of the logical language of the *Principia Mathematica*".[7]
>
> "As the logic had individual variables in its vocabulary, so the world would contain a variety of particulars, the names of which would be constants to replace, as extra-logical vocabulary, these variables; as the logic required only extentional truth-functional connectives between its elementary propositions, so the world would consist of independent extentionally connected facts; as the techniques of logic could define and thus make theoretically superfluous the more complex and abstruse concepts of mathematics, so, by the application of the same techniques, the less concrete items of the furniture of heaven and earth, the Meinongian underworld, could be defined and theoretically eliminated. The structure of the world would thus resemble the structure of the *Principia Mathematica*".[8]

7. Urmson, *Ibid* p. 16

The metaphysics of logical atomism, thus, appears as a justification for the truth-functional logic of Russell.

Has not logic—here, if not in Leibnitz, become the very life and breath of philosophy? Does not logical atomism clearly show that the kind of metaphysics which we adopt is *exclusively* determined by the kind of logic we accept? The answer is that it is indeed true that logic has determined the character of metaphysics at least partially in Leibnitz and fully in logical atomism. Yet, we must maintain that this does not have any tendency to show that logic has any essential or special role to play in metaphysics, except in the manner indicated above. For, in its endeavour to construct a picture of reality, metaphysics usually depends on some model or other. The model may be borrowed from any source. Thus, for instance, the Pythagoreans tried to give an account of the world in terms of their geometry while Locke seems to have modelled his metaphysics on atomistic mechanics. Some modern metaphysicians have borrowed the concepts of evolutionary biology. The fact that the model used may come from any source thus shows that its relation to metaphysics is an accidental or an external one. That it happened to be logic in the case of Leibnitz and of logical

8. *Ibid*, p. 7

atomism does not bring logic any nearer to metaphysics than if it were, say, archaeology, or cybernetics. As a matter of fact, later critics have pointed out that the logical doctrines of logical atomism do not imply its metaphysics but are independent of it. Our conclusion, then, that logic does not make any significant contribution to metaphysics, remains unaffected. One metaphysician may choose one model and someone else, another. For, in all metaphysics there is, of necessity, a subjective element.[9]

Therefore, if our conclusion regarding the relationship between logic and metaphysics holds in spite of this objection, it would be wrong to say, as Das does, that logic necessarily depends on, or implies, metaphysics. Das expresses the relation between logic and philosophy in two ways:

1. Understanding philosophy as the activity of critical reflection, he compares the relation between the two to that between genus and species; Logic, being a particular kind of critical reflection, is species, while philosophy, embracing other types of critical reflection as well, is genus. Species implies genus but not vice versa. "Philosophy does not imply, nor does it need, logic.'"[10]

2. Philosophy and logic may both be regarded also as the product of critical reflection. Thus understood also logic implies metaphysics since the concept of truth, the attainment of which is the aim of logic, implies and is determined by the concept of reality which is the object of metaphysics.[11]

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9. 'Is Philosophy Logic' in *Philosophical Quarterly* October 1962. This article led a symposium at the 37th session of the Indian Philosophical Congress held at Chandigarh in 1962. The same point is made by Dr. R. Das also in another article published earlier in the *Journal of the Academy of Philosophy*, Calcutta, July 1961.

10. *Ibid* p. 139.

11. *Ibid* Pp. 133-40.

Now, this view of Das creates considerable difficulties especially in view of his thesis that metaphysics is not open to attack either from without or from within and that in constructing a metaphysics, the philosopher follows his own and not anybody else's logic—a thesis which renders logic itself subjective and makes all discourse impossible. It is not the intention of this paper to examine in detail the contents of Das's paper. That has been partly done by his co-symposiasts Mishra[12] and Biswas.[13]. But it is worth noting that it is precisely because of the *kind* of things that Das says regarding the nature of philosophy that people have been led to revise their concept of philosophy, refusing to identify it with metaphysics.

The most violent attack on, and a total rejection of, metaphysics, came, of course, from the logical empiricists, who brought down philosophy from the starry heavens to mundane matters. Philosophy, for them, consisted simply in the analysis and clarification of language, not in constructing systems of metaphysics purporting to describe ultimate reality. This view has enjoyed considerable popularity in recent times, and has come to be regarded as an altogether different type of philosophy. It is, therefore, important for us to consider the relationship between it and logic. But before doing that, let us note a few of its basic features.

Devising a new criterion of meaningfulness—the so-called principle of verification—logical positivism summarily dismisses all metaphysical assertions as nonsensical and claims that far from revealing reality such assertions actually confound us. Reality is revealed, in so far as it is revealed at all, by the various sciences and we must turn our attention to them if we want to pursue this purpose, as the scientists do. There is no super-science,

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12. N. Mishra, 'Is Philosophy Logic'? *Philosophical Quarterly*—October 1962.

13. S.C. Biswas, "Is Philosophy Logic'? *Philosophical Quarterly*—January 1963.

alongside of the ordinary sciences, which is privileged to deal with the ultimate reality. The problems of metaphysics all arise from the misuse of language and they will all evaporate as soon as we rectify this misuse. Thus, the principal theses of logical empiricism are:

1. the rejection of metaphysics,

2. the principle of verification that the meaning of a proposition is the method of its verification, and

3. identification of philosophy with the analysis of language.

Now what is the nature of analysis with which the logical positivists identify philosophy? Curiously enough, they borrowed their conception of analysis from a certain school of metaphysicians who were their sworn enemies. These were the logical atomists whom we have already considered. As noted earlier, the logical atomists postulated a structural relationship between language and reality. Such a relationship is accurately expressed by logic, which, for the logical atomists, is extensional and which represents the syntax of the perfect language without the extra-logical vocabulary. But natural languages are imperfect and often their expressions conceal, instead of correctly projecting, the character of reality. It is these misleading expressions that give rise to conceptual confusions which are the basis of all philosophical problems. The point and purpose of analysis is to eliminate such misleading expressions by means of others which correctly picture facts. The task of analysis is achieved as soon as the structural relationship between language and reality is restored, and thereby the relevant philosophical problems are solved or, rather, dissolved. This is what the theory of descriptions and the theory of incomplete symbols (that is, the theory of logical constructions) were intended to do. It is thus that the so-called definite and indefinite descriptions and incomplete symbols were eliminated in favour of basic propositions which unambiguously pictured facts. It is thus also that the

shadow world of Meinong and the different universes of discourse were disposed off. In the words of Wisdom:

> "The theories of descriptions and numbers, and the theory of classes were all pieces of logical analysis and they worked like charms on many hitherto incurable philosophical complaints. The proof of the unreality of space, the ontological argument for the existence of God and the extra entities in the universes of discourse, all went up in smoke, though from the fictional entities there lingered still a peculiar smell."[14]

Though the procedure of elimination involved both same-level and new-level (or reductive) analysis, the former was regarded merely as a preliminary to the latter. For, it was ultimately the latter which brought out the parallelism between language and reality and therefore the final aim of the atomists was reductive analysis, and the rationale of such an analysis was the atomistic metaphysics.

Now the logical empiricists, while they rejected the metaphysics of the atomists, bodily took over their notion of analysis. When confronted with the question of the justification of their analysis they retorted by simply denying the problem of justification. They held that the function of analysis is to reveal, not reality, but the logical structure of the language of science, and of informative discourse. This was in line with their anti-metaphysical stand. They also conceived language in the same manner, that is, as extensional and truth-functional. Because the activity of constructing accounts of reality was rejected, the only thing that remained for the philosopher was analysis and clarification of language. Philosophy, thus, became identical with analysis, unlike what it was earlier in logical atomism.

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14. John Wisdom in his essay 'Metaphysics and Verification; *Philosophy and Psycho—Analysis.* Oxford, Basil Blackwell—1953, p. 59

Let us now return to our main question. If this is the character of analysis with which philosophy is identified, how is logic related to philosophy? The answer is that philosophy is not different from logic, it simply *is* logic. The propositions of philosophy, like all others, had to be either empirical or *a priori* according to the verification principle, and since they obviously could not be empirical, they were regarded by Wittgenstein and other atomists as *a priori*. The propositions of philosophy being instances of analysis, state equivalences between expressions, (which is the form which analysis takes). Hence they are tautologies, true for all truth-conditions. They can also be regarded as definitions, since in a definition the definiendum and the definiens are equivalent. As definitions and tautologies are the concern of logic, in doing philosophy we are doing no more than logic. This, of course, is not to say that logic and philosophy are identical, as some extremists hold. For, in logic we do many other things also besides considering definitions (e.g. considering implication-statements, and the different relations between propositions). Logic is thus wider than philosophy. Its relationship to philosophy is like that of the whole to its parts. Or, reversing the terminology of Das, we may say that logic is the genus, philosophy, the species. The species implies the genus, not *vice versa*.

This conclusion that philosophy is logic follows if philosophy is regarded as identical with analysis. But the question is whether it can be so regarded. Very many reasons have been brought forward for saying that philosophy is not analysis. Of these, we state below a few typical ones.

1. The method of reductive analysis may have been applied with some success in mathematics, but outside of mathematics, analysis achieved no success worth the name. For instance, nation-statements could not be reduced to individual-statements and physical

object-statements could not be reduced to sensation-statements.[15]

2. Reductive analysis gave the impression that the philosopher eliminated what he analysed. When nations were sought to be reduced to individuals, they were thought to disappear from the world like chimeras and numbers. But no such elimination is as a matter of fact possible. When Russell analyses : "Two men were there" into "A man was there and another man was there," the notion of number is not metaphysically eliminated. The word "Other than" which Russell employs in such analyses already implies plurality, that is, number. Even if such an elimination could be achieved, it goes against the very claim of reductive analysis, that it is expressed in the form of an equivalence. If one side of the equivalence is eliminated, how can there be equivalence at all? This is what Wisdom calls "The Paradox of Analysis".[16]

3. The view that philosophy is analysis involves the assumption that the only purpose of indicative sentences is to state facts; otherwise, the verification principle would become inapplicable and would not be able to dismiss metaphysical, ethical and other statements as meaningless, though they are in the indicative form. But such an assumption is obviously false.

These are but a few of the numerous difficulties in the view that philosophy is analysis. If philosophy cannot be regarded as analysis, the question—what is philosophy?—reasserts itself. The difficulties that we have noticed so far show that philosophy is neither metaphysics nor analysis. To avoid these difficulties, we have to say, as Wisdom[17] and Wittgenstein[18] do, that philosophy

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15. John Wisdom, *Ibid*, p. 60.
16. *Ibid*, Pp. 65-66.
17. *Ibid.*
18. L. Wittgenstein—*Philosophical Investigations.* Oxford, Basil Blackwell, 1953.

is not analysis or definition but a description of the use of our expressions. The meaning of a proposition is not the method of its verification but the method of its use. Says Wisdom.

> "Metaphysics is not analysis......it is better to describe...metaphysics......as sorts of game played with words as pieces......and then to define metaphysical......sentences by reference to the purposes they serve in the game."[19]

The philosopher is concerned with such questions as : Are X-facts identical with Y-facts? Usually there is no right or wrong answer to these questions, though sometimes there is a correct answer 'No'. But in either case the metaphysical problem is resolved by explaining what induces each disputant to say what he does. This is done, firstly, by explaining the nature of the question and, secondly, by providing the required descriptions.

Wisdom is expressing essentialy the same views as Wittgenstein does in his *Philosophical Investigations.* Wittgenstein, rejecting the conception of a uniform and rigid language having a single function, as presented in his *Tractatus*, now declares that language has no such uniform character and no single use. He says,

> "Think of tools in a tool box....the functions of words are as diverse as the functions of these objects"[20] "How many kinds of sentences are there? There are countless kinds......and this multiplicity is not something fixed....It is interesting to compare the multiplicity of the tools in language and of the ways they are used, the multiplicity of kinds of word and sentence with what logicians have said about the structure of language. (Including the author of the *Tractatus*.)"[21]

19. Wisdom. *op. cit.* p. 61.
20. Wittgenstein—*Philosophical Investigations.* Para 11.
21. *Ibid*, para 23.

It is in vain to say that behind all the diverse uses there *must* be something common because there is nothing in common.

> "What is common to them all? Don't say : there must be something common, but look and see....If you look......you will not see something that is common to all, but similarities, relationships and a whole series of them at that."[22]

This means not merely that words do not get their meaning in one single way, but also that in thinking so we are labouring under a superstition.

> "It is like a pair of glasses through which we see whatever we look at. It never occurs to us to take them off."[23]

How does one get this superstition? Its source is language itself. The superstitious beliefs are produced by

> "the bewitchment of our intelligence by means of language"

or

> "through a mis-interpretation of our forms of language."[24]

Therefore, we must look

> "into the working of our language and in such a way as to make us recognise those workings in spite of an urge to misunderstand them. We must try to command a clearer view."

The bewitchment by language consists in the fact that we tend to apply certain stereotype questions and answers, which apply to a large number of cases of a certain kind, to cases which appear to fall within the same kind but which really do not.

> "It is as if the surface of our language were thickly

22. *Ibid*, para 66.
23. *Ibid*, para 103.
24. *Ibid*, para 109-110.

covered with well-trodden paths and we were constantly tempted to follow these paths even when they did not lead in the direction we were trying to go."

In other words, a picture holds us captive.[25]

Philosophical problems thus arise, according to Wittgenstein, from the misleading character of our language. They have a characteristic unclarity and power to confuse and have the form "I do not know my way about." It is not that they result from an ignorance of facts or from strictly logical errors in reasoning. They arise from a fundamental confusion about the use of the relevant expressions. If so, the solution consists not in thinking but in looking into the actual uses of the expressions and in describing these uses.

> "We must do away with all explanation, and description alone must take its place."[26]

These uses are already there and lie open to view and what we do is to bring words from their metaphysical to their everyday use.

This does not mean that philosophical problems are trivial; they have the character of depth. Their roots are deep in the forms of our language and they are of great significance. Nor does this mean that philosophical problems are verbal. Though they are about language, they spring from language. They may equally well be said to be about concepts. Wittgenstein, when he talked of philosophical problems as being about language, meant by language not something verbal or an arbitrary system of conventions. Language for him was a form of life. "To imagine a language," he said, "means to imagine a form of life."[27] For him language and reality and thought are not different from but inextricably connected

25. *Ibid*, para 115.
26. *Ibid*, para 115.
27. *Ibid*, para 109.

with each other. They are different aspects of one and the same thing. Philosophy in dealing with forms of language, therefore, is also dealing with forms of our life, forms of reality and thought. It is in this sense descriptive of reality, and not verbal as the logical positivists held.

If this is the view of philosophy that we have to take, then what is our answer to the problem of the relationship between logic and philosophy? The answer is this: philosophy cannot be logic because it is not analysis. It does not consist in giving definitions such that the definiens and the definiendum are equivalent. It consists in the description of the uses of our expressions and such a description is also a description of reality. But it is not a description of the detailed nature of reality as is to be found in the sciences. It is a description of the formal features of reality. When we are considering the use of an expression we are not interested in that expression as such; we are interested in it as an instance of a certain type or form of use, though instances of such form are rare and the form itself is not clear-cut. Thus, in dealing with philosophical questions, we are concerned with essentially general enquiries. This is also indicated by Ryle's statement that philosophical puzzles are category-mistakes; for categorial statements are formal and general. And the form they express is not something ideal or normative. It is already present in the actual use of our language. The philosopher simply draws our attention to it. If so, philosophy has certain things in common with logic. Firstly, logic is also concerned with forms and its forms are also actually present in our reasoning. When logic is described as normative, it is not intended to be like ethics. In ethics the ideal or the form is not always a description of what already exists. It may be something new. There may be, in other words, a gap between the "ought" and the "is." In most cases the "ought" may be a formalization of the "is" but there may be cases, especially in the frontier regions, where totally new standards, which had never been previously applied or used, may be laid down. For example, we may evolve a code of space-ethics and this code may demand patterns of behaviour which in our actual life we never previously practised. When, therefore, logic is said to be normative, it is simply intended to draw our attention to the fact that certain standards are, all of them, actually present in our thought. Thus, the questions of logic and philosophy are both formal and general.

Secondly, just as the propositions of philosophy are not verbal, the propositions of logic also are not verbal. When, for example, we say, "A cannot both be and not be B" or "If A is equal to B and B is equal to C, then A is equal to C," we are not making statements that merely record our determination to use symbols in a particular manner. Such statements are necessarily true, not because we arbitrarily decide to use words in the manner in which we actually do. That may be true of such assertions as "An oculist is an eye-doctor" or "A bachelor is an unmarried man." Such assertions may be said to be true because of the meaning of their words, and therefore, they would really be disguised empirical propositions. We have to look around to see whether they are true, only the looking around here consists in looking into the dictionary. (Even in such cases, there are people like David Mitchell, who would maintain that the statements are necessarily true and their necessity has nothing to do with the words used in them.) But typically logical propositions are topic-neutral and are independent of the meaning of their words. They are true because of their forms and the forms, once again, are the forms of reality. It would, therefore, be true, though paradoxical, to say that what makes the propositions of logic true is reality itself. But that is what we say of empirical propositions. How do the two differ, then? In this, that while what validates the logical propositions is the general or universal aspect of reality, what validates

empirical propositions is its detailed contents. The same thing can be said of philosophy. The propositions of philosophy, as we have noted, are general and formal. The philosopher, therefore, is not making any trivial statements; his propositions are profound in the sense of describing formal features of reality, though not in the traditional metaphysical sense.

It is important to note these similarities between logic and philosophy, but it is equally important to note the differences between the two. For, obviously the two are not the same. If we argue from these similarities to the identity between logic and philosophy, then we commit the typical philosophical error against which Wittgenstein warned us. We will be swayed by the lure of a model of a picture, by the bewitchment of language. The difference of philosophy from logic consists, firstly, in that while the forms of logic are precise and well-articulated, the forms of philosophy are mobile and flexible; and, secondly, in that while the forms of logic do not cause any perplexity and are easily identifiable, the forms of philosophy generate considerable puzzlement and are revealed only after very careful and prolonged thought.[28]

Borrowing an analogy from Ryle, we may say that while the activity of the logician is like a parade-ground drill, that of a philosopher is like the fighting on a battle-front. (Ryle's other analogies are logic is to philosophy as geometry is to the cartographer, or accountancy is to the merchant or the exchange power of a coin or legal tender is to the behaviour of a consumer commodity.)[29] Though fighting and parade-ground drill are vastly different, yet an efficient and resourceful fighter is also a well-drilled soldier. The moves and manoeuvres on the

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28. Gilbert Ryle—*Dilemmas.* Cambridge University Press, 1954. P. 118.
29. Ibid, Pp. 119—21 and 123.

parade-ground are limited, while those in the battle are unlimited and extremely various. Yet they spring from the former. The formal rigours of logic can never be found in philosophy. The problems, say, in the elucidation of the concept of pleasure or of perception or of freedom are not such as result in contradictions when the relevant statements are denied. They become clearer as the discussion progresses and whether a given philosophical argument is valid or not is itself a debatable point. Generally, the question is whether the argument has much or little force. Yet there remains a very important sense in which the enquiries of both philosophy and logic are 'logical'. The logician works out the logical powers of his topic-neutral logical constants and the philosopher explores the logical powers of certain concepts which are not topic-neutral. The considerations which are decisive for both are logical, just as in the choice of drill-evolutions and in the choice of battle evolutions the considerations that are decisive are military considerations.[30]

But if what we have said is correct, what happens to the difficulty that Wittgenstein himself posed in the following passage? " 'Sentence' and 'language' have not the formal unity that I imagined, but are families of structures more or less related to one another. But what becomes of logic now? Its rigour seems to give way here. But in that case does not logic altogether disappear?"[31] The answer is, as can be gathered from the foregoing pages, that far from disappearing, logic remains very much alive, and alive in its own undiluted rigorous form. A calculus has as much precision and rigour as its authors give it, and certainly no damage is done to the rigour of logic by the discovery that a natural language has less. For, as we have seen, the logical constants,

30. Ibid, Pp. 118-9.

31. Wittgenstein, *Philosophical Investigations*, para 108.

whose inference powers the logician investigates, are not merely topic-neutral, but also conscript-terms under military discipline. The territory of logic, therefore, is completely charted and mapped. But the concepts in natural languages whose logical powers the philosopher investigates are neither topic-neutral nor colourless. The territory of language is in many ways not only uncharted but can never be charted. Hence language can never have the formal rigour of logic and there is no question of its losing that rigour.

We may conclude with a reference to what P.F. Strawson says in this regard.[32] Distinguishing between entailment-rules and referring-rules, Strawson rightly says that the logician is concerned with the entailment rules since his statements are abstracted from all contextual setting. Referring rules lay down the contextual requirement for the correct employment of an expression.[33] As such, they cannot enter into the logician's discourse. Strawson further observes, again rightly, that type-rules cannot be assimilated to formation-rules on the ground that to deny them is not self-contradictory but nonsensical. For example, the expression, "The cube root of 10 is 10 miles away" is not so much self-contradictory as senseless. While referring rules may not be the concern of the logician, they certainly are the concern of the philosopher, who describes the uses of linguistic expressions. In spite of this distinction, the study of formal logic and the study of the logical features of language are overlapping. They are also mutually illuminating. Only, the simple deductive relationships are not the only relevant relationships needed in the study of the logic of language. We have to think in more dimensions and use many more tools of analysis

32. P.F. Strawson, *Introduction to Logical Theory.* London, Methuen and Co., 1952, Especially the Chapter on 'The two kinds of Logic.'

33. Ibid. p. 213.

than are found in the study of logic.[34]

While we agree with Strawson in all this, we do differ from him in a very fundamental respect: Strawson says that all necessary propositions—both formal and non-formal—are of linguistic origin. He says:

> "Rules about words lie behind all statements of logical appraisal."[35]

But, as we have remarked, at least formal necessary propositions, that is propositions of logic, are independent of linguistic considerations.

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34. *Ibid*, Pp. 231-32
35. *Ibid*, p. 36

Y.D. Shalya

ANALYTICITY

ORDINARILY, the truth of a statement depends upon or is a function of the state of affairs it refers to. For example, the truth of the statement "White pearls are piled in heaps on the moon" is dependent on the situation referred to therein and is true or false *only* in the context of that situation. But there are some other statements which, though ostensibly seeming to refer to some state of affairs, do not in fact refer to it in the same way as the above statement does. For example, statements such as "Malarial germs are killed by quinine" or "Cosmic rays induce many a time the process of gene-mutation" or "Light rays are propagated in space both as waves and particles" appear to claim something about certain states of affairs and, thus, if they actually happen to be such, the statements would thereby become true. But, strictly speaking, "gene-mutation" and "malarial germs" and "light propagation" are theoretic constructions and they are not in the world in the same sense as "tables" or "my pain." They may also be understood as imaginary pictures, such as "germ-picture" or "particle picture." In other words, these statements express conceptual constructions. As an example, we may take the germ theory of disease. These germs are to account for the disease in its origin, spread and cure. In fact, it is only in the context of such experiences that they could be successfully established or refuted. But it is because of this very reason that the statements formulating the generalized principle of the germ theory of disease may hardly ever be refuted. Germs, for example, are a completely irrelevant hypothesis in the conceptual construction of other systems of medicine, such as homoeopathy or Ayurveda. The two images of the propagation of light rays in the conceptual scheme of modern physics can be seen as another illuminating example of it. In this way, statements which refer to conceptual constructions, or rather which contain them as an essential constituent, may be established or falsified in different ways. In reality, to call them as either true or false is more misleading than otherwise. Rather, such statements, made in the essential context of generalised principles, should better be termed as "appropriate" or "inappropriate," since they cannot be verified as true in terms of any definite experience at all. As for being falsified, this can hardly even happen to them in any meaningful sense of the term at all. Even so, experience is relevant to them in some sense or another. But there are two other types of statements for which experience is completely irrelevant. These are called logical and analytical statements, respectively.

Logical statements are considered true or false only on the basis of the connectives and the values of the variables. For example, logical statements such as $(x)\ (px \supset qx)$ or $(x)\ (px \supset \sim qx)$ depend for their truth or falsity on the accepted usage of the connective signs $\supset$, $\sim$ and the truth values of the variables concerned. No specific meaning is required for p and q in these statements nor is any verification relevant here. However, if descriptive terms are substituted for these variables, then the resulting statements that we get are called analytical statements. These statements are true just by virtue of the very meaning that they have. For example, if in the first statement we substitute 'bachelor' for 'p' and 'unmarried' for 'q', we get statements which are true by virtue of their meanings themselves. In other words, statements built out of descriptive terms, which are true independently of the state of

affairs described or referred to therein, are 'analytic statements'.

Now, there seems little dispute about this definition. However, the real issue seems to centre around the question whether there actually are any such statements or or not. First, a statement's being true by virtue of its very meaning and its being true independent of the situation it refers to, is not one and the same thing. Secondly, there are great difficulties on either of the formulations. For example, what exactly is meant by the meaning-grounded truth of such a statement as (e) "All bachelors are unmarried"? Now this can mean at least two things:

1. The term "bachelor" and the term "unmarried" have completely identical meanings.

2. If any person is married, he cannot be a bachelor while if he is unmarried, then he must also be a bachelor.

Now, in an extensional language these two statements are merely two different ways of saying the same thing, while in an intensional language they are really two different statements. To assert the identity of meaning of two statements in an extensional language is merely to state that they point or apply to the same object or objects in the world; and this is exactly what has been stated in the second statement. As against this, in an intensional language, the second statement will be empirical rather than analytic, as it states that the class of persons denoted by the term 'married people' excludes the class of people denoted by the term 'bachelor'. But, remaining within the limits of extension or denotation, how can we know that the members of the class of 'bachelors' and the members of the class of 'unmarried persons' are the same or identical? In case, it is by decision, then instead of explicating what is exactly meant by identity of meaning and then analyticity, we would be rather *presupposing* them. On the other hand, if it is said that we know the identity of meaning on the basis of investigation or experience, then the statement will no more be analytic but synthetic.

Besides this, in a purely extensional language two heteronomous terms would have to be held as synonymous. For example, any term which designates a null class would become necessarily identical in meaning in an extensional language, say, such as "Shiva" and "Rahu." But no one would be prepared to admit that the two are the same just on this ground. The usual reply, of course, is that it is only in the context of mythological stories that they are not extensionally identical. Professor Goodman[1] has presented many such arguments in defending the purely extensional theory of meaning but, as we have shown elsewhere,[2] these arguments are not satisfactory and that the identity of meaning is possible only in an intensional language.

The question of the identity or equivalence of meaning is, however, not easy in an intensional language either. First, the problem arises as to what exactly do we mean by "intensional meaning?" Is it to be understood in a behaviouristic sense or in an intuitionistic one? But, for the present, we may postpone this discussion and accept the definition of identity of meaning, following Quine, such that two statements are identical in meaning when these "two sentences command assent concomitantly and dissent concomitantly, and this concomitance is due strictly to word usage rather than to how things happen in the world."[3]

1. Nelson Goodman—"On Likeness of Meaning," in Leonard Linsky (Ed.) *Semantics and the Philosophy of Language*, The University of Illinois Press, Urbana, 1952.

2. Yashdeo Shalya—*Darshanika Vishleshana*, Pp. 160-76, Akhil Bharatiya Darshan Parishad, 1962.

3. W.V.O. Quine—*Word and Object*, p. 62, The Massachusettes Institute of Technology, 1960.

But, then, how to know that two terms or sentences have the same meaning in a given language? In other words, how to know that all the speakers of a language L accept or reject the said words or sentences in the same way? In such a situation, the statement "these sentences have the same meaning in L" would only mean that all those who use the language L accept or reject p if and only if they accept or reject q also. But, then, this would be a statement about the identity of meaning and therefore a synthetic statement and not analytic in character. For example, instead of saying " 'All bachelors are unmarried' is analytic in the language L" we would say "All persons who use the language L accept the statement that 'all bachelors are unmarried,' " which, obviously, is synthetic.

This difficulty, however, may be solved in another way. It may be urged that for me the terms 'bachelor' and 'unmarried' have the same meaning and thus, the above sentence is analytic for me. But such a definition of identity of meaning and, thus, of analyticity would only please the opponent who would argue that this makes the concept completely arbitrary and useless for any purpose. It will make any sentence analytic, provided we are prepared to hold it "come what may." However, such an objection cannot really be made against what we are saying: for what we are asserting is not that we decide to accept the statement 'p' as analytic but only that I find it to be such and that this is due to my being a member of the class of those who use the language L. In case any individual using the language L says that he does not accept the above statement, then we will only say that he accepts the use of the term 'bachelor' and 'unmarried' in a different manner from the one I and other users of the language L accept.

Such an interpretation of the situation brings us in direct opposition to the philosophers who contend that analytic statements are merely postulational statements established according to the rules decided upon by us.

One group of such philosophers claims that such statements are used in ordinary language also, while the other group denies that such *decisions* are ever found in ordinary language and thus denies the relevance and usefulness of the concept of analyticity in that context. But even if we assume that a linguistic rule is established by decision, then this decision will have to be established through other sentences and the meaning of these sentences establishing the rules will only be obtained by intuition unless we want to go one step further down where in any case we will have to obtain it ultimately that way. Thus, ultimately, we have to accept intuition in the context of meaning if we do not want to be involved in an infinite regress. But, then, there should be no difficulty in accepting the view that we get the analyticity of the analytic statements also through intuition.

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If analytic statements are not found in ordinary language, then they will not be found anywhere else except in formal linguistic systems, for viewed in this context, scientific and philosophical languages are no better than ordinary language. Ordinary statements, which are clearly definitional such as "All teachers teach" or "All bachelors are unmarried" are neither used in any ordinary language nor in any special language. They are merely, so to say, ideal sentences. In actual use, whether it be in an ordinary or in a special language, only statements which are implicit definitions occur properly, such as "Nothing can be at two places at the same time" or "Nothing can be wholly green and red at the same time." These also occur in such cases where the usage is sharpened or made exact in a special context, for example, "Man is a rational animal" or "I alone can experience my own pain."

Implicit definitions and accepted usages in specific or determinate contexts are found at all levels of linguis-

tic discourse. However, Morton White[1] objects that any such statement with respect to ordinary language always reflects an arbitrary decision on our part and not of the way it is actually used in that language. According to him, we can formulate a rule in an artificial language such that in all contexts the term "man" is replaceable by the term "rational animal." In this artificial language L_1, though there is such a term as "featherless biped" in its vocabulary, there is no such rule that it can be substituted for the term "Man" in all contexts. In this language, then, the statement "Man is a rational animal" will be analytic, while the statement "Man is a featherless biped" would be synthetic. As against this, we can build another artificial language L_2 in which the statement "Man is a featherless biped" becomes analytic and the statement "Man is a rational animal" synthetic. Now, according to White, the question which statement is analytic in language L_1 or L_2 can easily be decided by looking at the rules of these two languages. But there is no such set of rules in ordinary language which we may call L_3. How to know, then, that in this language which of the two statements, which are both in use there, is analytic or synthetic? There is just no clear-cut list of rules there. In such a situation, which of the two languages L_1 or L_2 may we treat as a logical reconstruction of L_3 and which not?

White is obviously right in making this objection, but not from the viewpoint which he wishes to establish. Rather, the objection can be made sense of only from a viewpoint completely opposed to his. Both the statements, in what White calls ordinary language, may become analytic depending upon the decision of its speakers or upon the contexts in which they are spoken. This is what is meant by its being undecided in this

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1. Morton White—"The Analytic and the Synthetic," in Linsky (Ed.) *Semantics and the Philosophy of Language*, p. 279.

respect. But while White wants to consider it as intrinsically undecided in this respect, he also wants to treat it as if it were determinate and decided and had specific rules of its own. But L_3 is the language which essentially comprises within itself different uses by different persons and also different uses by the same person in different contexts. To borrow the analogy from White, it is like a herd of black goats and white sheep in which the two get prominence at different times. This ordinary language is *actually used* in such a way that sometimes the language L_1 may correctly be regarded as its logical reconstruction and sometimes the language L_2. The trouble with White is that though he argues from the indeterminacy of the language L_3, he presents it as if it were a determinate language. Otherwise, it is meaningless to ask as he does, "which is the logical reconstruction of L_3, the language L_1 or the language L_2?." The answer, obviously, is both.

However, we cannot accept even this contention that analytic statements are established purely through decision. This is true only of postulational languages, but not so even of metaphysical reconstructions or of philosophical analyses. For example, when we say that (A)—"if anything is red all over at time t_1—t_2, then it cannot be green all over at t_1—t_2", then it will not be correct to say that "the sentence (A) is absolutely assertible in language L_1", for this is a rule and not an assertion. It will be more correct to say that (B)—"the absolute assertibility of the sentence (A) in language L_1 is a rule of the language L_1", which is a description. In contrast to this, even when we try to determine a sentence in a context we are only trying to clarify and bring into light the implicit conceptual scheme involved therein.

It may be objected that it would be unwarranted to call the sentence here as analytic, since it is as much a statement of a state of affairs as any descriptive statement ever could be. This obviously is due to the

simple reason that it is falsifiable by relevant experience. This, though correct, does not make (A) a synthetic statement; it is only the statement (B) that becomes such.

The real difficulty concerns the unconditional assertibility of the statement (A). We never use an unconditionally assertible statement in a situation where the hearer knows its use already. For example, it is unlikely that a philosopher will say to another philosopher "All material things are extended." As against this, however, if we tell this to a child in order to give him some information, then it does not remain analytic any more. In that case, the statement (C) *"a is a material object ≡ a is extended"* becomes synthetic, for the child does not know the relation between these two statements. The point of this objection is that because the relation of equivalence does not obtain between the statements for the child, therefore, to him, the second statement does not merely make explicit the meaning of the first statement. But, then, the question arises as to what does it exactly do? The statement "All the marbles in this bag are green", for example, gives some information. Now is this information of the same type as the one given by the statement (C)? Obviously, there is a great deal of difference between the two. The information given through the statement (C) is that its component statements are related in an analytic way. The objection, on the other hand, may be taken to mean that while the relation of biconditionality asserted in the statement (C) does not exist for the child, therefore the compound statement is not analytic for him either. The information that we give him, then, is that "in the language L, the statement 'a is a material object' and the statement 'a is extended' are related by "≡" and this is as good an information as the one given by "All the marbles in this bag are green." Certainly, these two informations are of the same type, but, as we said earlier, this does not make the statement (A) itself synthetic; what really becomes synthetic thereby is the statement that *"the statement (A) is unconditionally assertible in the language L_1"*

* * *

Analyticity is a concept belonging strictly to a language. A statement is analytic only if its truth or falsity is a function of the meaning of its constituent terms. Or, in other words, it does not depend for the establishment of its truth or falsity on the knowledge of any event or state of affairs in the world. Some philosophers think it proper, on this ground, to call all analytic statements vacuous. There could be two meanings to such a proposal. Take, for example, the same old statement "All bachelors are unmarried." Apparently, this statement seems like the statement (D) *"All men are mammals."* But, in fact, this is not the case. The logical form of (D) is that "the class of men is included in the class of mammals" and it is difficult to say whether the meaning of the term "mammalian" is conceived as an essential part of the meaning of the term 'man'. As against this the terms 'bachelor' and 'unmarried' are completely equivalent in meaning and so replaceable in all sentences without effecting their truth value.

Accepting, thus, the doctrine of complete replaceability we get sentence (E) "All bachelors are bachelors" by replacing 'unmarried' by 'bachelor' in "All bachelors are unmarried." This is what is meant by the vacuousness of the statement "All Bachelors are unmarried." But sometimes the vacuousness is not so directly obvious, especially where it is due to some specific conceptual scheme that we happen to have adopted and which alone has resulted in making the statements vacuous in character. For example, the statement (F) "I alone can experience my pain" may not appear directly vacuous to many persons. Yet, in the context of a conceptual scheme, it may really be so.

Ordinarily, for example, we do use the terms 'sympathy' and 'sympathise' in such a way that the statement

"I sympathise with Rama" may be reframed as "I share the pain of Rama." Thus, the statement (F) appears to give us some new information, but, as is obvious, the difference can only be either conceptual or linguistic in character. It could be regarded as conceptual only if we show that we have such a concept of pain that to say "I am experiencing another's pain" is a mistake. But, linguistically, it can be seen this way: Pain is an experience and experience is only singular; it cannot be multiple; thus, the pain of Rama and the pain of Shyama, which are two experiences, cannot be held to be just one single experience. Established in this manner, the statement "No one else can experience my pain" becomes just a tautology.

But the above linguistic interpretation may be challenged on the basis of a reinterpretation of the concept itself. How far, it may be asked, is it meaningful to attach the adjective "one" to things such as "experience"? Can one really call "experience" as "one" in the same sense as we call a chair or table "one"? This is not clear at all. Perhaps, the use of the adjective "one" for "experience" is not quite improper though certainly its use here is not as indubitably correct as when it is used in respect of a chair or a table. Further, it has also to be shown that Rama and Shyama cannot jointly experience the same identical experience. At least, the possibility of co-experiencing cannot be ruled out by itself. One may, of course, try to deny it on the ground that "experience" is an "event" that is "enjoyed" and thus the pain of Rama is enjoyed by Rama and the pain of Shyama enjoyed by Shyama. These experiences are different events and thus they cannot be absolutely identical. Against this, it may he argued that, even if it is admitted that the "enjoyed experiences" of Rama and Shyama are two events, why should we not regard the "different enjoyment events" as different events of the same one identical experience? Why should they be regarded as different experiences? In other words, why should not the statement "Rama and Shyama share this pain" be reformulated as "the pain-event of Rama and the pain-event of Shyama are two occurrences of the same pain?"

In such a situation, there seems really no incontrovertible reason to prefer one way to another, as what we are really doing here concerns only the decision whether we would call the sympathetic pain-experiences of Rama and Shyama as two different experiences or as two different events of the same experience. Whatever the decision we may make, it will be only that decision which will make the above sentence analytic, or else unacceptable.

From the viewpoint of the ordinary common language, the sentences (E) and (F) are very different sentences. The ordinary language has clear and unambigous directions about the use of the terms "bachelor" and "unmarried" in the sentence "All bachelors are un-married." There is, however, no similar direction about the use of the terms in the sentence (F). It is, therefore, largely left to us how to decide the matter.

As is evident from the above analysis, the statement (F), though completely vacuous from the viewpoint of facts, is not so from the viewpoint of the concepts themselves. It is not a conceptually vacuous statement constructed out of terms which have already been decided to have a particular meaning context. Rather, it states in a conceptual context as to the perspective in which we will see such facts as those of sympathy or telepathy. The statement (A) that "Nothing can be green and red at the same time" is of the same type. We claim here that the analyticity of the statement is an essential part of the language we use to communicate about the external world. Now, if we see the blue-red sky of the morning, we discover that the same object or space is pervaded by these two colours at the same moment. If this is taken to deny the truth of the above statement, then obviously it is synthetic in character. But if we do not want to treat such facts as relevant to

the establishment of the truth or falsity of the above statement, then we can make another decision in the matter. We may, for example, accept blue-red as another colour and give it a different and distinct name of its own and this decision will be as convenient as anything else.

The purpose of the above rather detailed consideration is to clarify as to what actually is involved in the decision to change a linguistic usage and what its exact form is. When we decide, for example, to treat, "blue-red" as one colour or to treat the statement (A) as not analytic, we are not giving up the very basic meanings of the terms, but rather deciding about the particular conceptual scheme we wish to adopt in the matter. The occasion or the opportunity for such a decision concerning the conceptual scheme is derived from a lack of clarity in the ordinary usage on the one hand and the absence of any clear, determinate directive deriving from the state of affairs, on the other.

We have decided, for example, that we will treat the analyticity of (A) as the essential constituent of the meaning of terms denoting colours. But, as we have already said earlier, we could decide otherwise and the result of that decision would be sufficiently significant, as we would then be able to use the following two sentences simultaneously:

1. The Table M_1 is red all over at t_1.
2. The Table M_1 is green all over at t_1.

But it should be noted that no scheme will allow us to say:

3. Not (The Table M_1 is red all over at t_1) along with (1), for this would mean that this table is not red at all, which would be false. If the table M_1 is red at t_1, then none of its part can be devoid of the colour red at t_1. To accept the sentence 3 would be to deny the law of contradiction or to accept an exception to it.

Thus we find that even if analyticity of statements be considered a function of their meaning, what is included in this class is not merely the class of those statements which clearly and explicitly have the same meaning, but also the class of those statements which express implicit definitions and which often give us some insight into the state of affairs also. But all statements cannot be classified into analytic-synthetic. There is another class of statements which are true or false, not on grounds of their meaning alone or on grounds of the facts they refer to, but only in the involved context of some one or other conceptual scheme. Most of the philosophical or scientific statements are of this kind. The statement "Two parallel straight lines never meet," for example, is correctly assertible only within Euclidean geometry. In the context of other geometries, it is just false. Thus the truth or falsity of the above statement cannot be determined on the basis of the meaning of the term "straight line" alone. Rather, it is only in the context of the geometrical theory as a whole that we may judge it meaningfully. In Einstein's system, the statement "Two parallel straight lines will meet, if drawn to sufficient length" is true and the accepted truth of this shows that the meaning of the term "straight line" is determined in the context of a theory and not in independence of it. The truth of this has been shown by the acceptance of the particle and wave theories of light on the one hand and the rejection or refutation of Newton's theory of gravitation, on the other.

But, really speaking, we need not have waited for the Einsteinian revolution to show the truth of this assertion. In the field of philosophy it is usual for two philosophers to build philosophical constructions in the context of different conceptual systems. For example, the seemingly simple statement "There is a table in the next room" would mean differently to a philosopher who accepts the independent existence of physical objects and to one who believes only in the reality of sense-data.

The question of the truth of statements on grounds of what the constituent words mean, thus, does not arise either in the field of philosophical analysis or in the field of what is known as "scientific theory." For, the statement here is not made on grounds of the meaning of the terms that are used, but rather on the basis of that decision which the philosopher makes in the choice of his conceptual scheme. In such a conceptual scheme, he certainly constructs definitions, but these definitions cannot be regarded as analytic in the sense of the identity in meaning of the terms on the two sides of the equational sign.

To call them "synonymous" is to change the very meaning of "synonymity." Suppose, for example, we take the statement *"This thing before me is a table"* which describes the state of affairs prevailing at that time. In the sense-data conceptual scheme, it will be supposed to describe the following: "Some visual sensations are occurring now," while in the physical object conceptual scheme the same statement will be supposed to describe something like the following: "Some visual sensations are occurring now because of the table" or "Some visual sensations related to the table are occurring now."

Similarly, in a mystic or spiritualistic conceptual scheme the "experience of God" is a proof for the knowledge of God, while in an empiricist conceptual scheme, it is only a claim to a certain experience and not to the knowledge of the object of that experience. In astrology, similarly, the statements concerning future based on the position of the planets at the time of the birth of a person are cognitive propositions, while they just could not have any such meaning in a physicalistic framework.

Synthetic statements state something about extra-linguistic state of affairs. But this capacity of theirs is not quite free or independent of some conceptual scheme or other, though their truth or falsity is certainly not determined by the conceptual scheme they presuppose. For example, the statements concerning external objects, whether they are made in the context of a sense-data conceptual scheme or in a physical object conceptual scheme, will be verified in the same way. But their meanings will certainly be different in the two different contexts. The same will be true of the statements concerning mental events. However, this would not be true in the same way of statements that concern scientific principles, though such statements as "water boils at 100 degree centigrade" are verified in the way other synthetic statements are verified. But, on the other hand, statements such as "Light-waves propagate in quanta in space" or that "Planets revolve round the sun on account of its gravitational power" may neither be held to be synthetic like the statement "There is a table in the room nearby" nor analytic like the statements "A thing cannot be in two different places at the same time" or that "I alone can experience my own pain." Rather, these are statements of a different sort. We may call them explanatory, if we so like, for they do not describe any event but occur rather in the context of their explanation.

There are still another kind of statements which are regarded as analytic. These are the statements denying the truth of contradictory statements, such as, for example:

(Rama is eating and not eating at time t_1...tn.)

But some thinkers have argued for the acceptance of self-contradictory statements as true in certain contexts or conditions. A.J. Ayer[1], for example, considers the law of contradiction as merely a matter of convention. According to him, if we change the convention regarding the use of the term "not," then we can use the contradictory statements. Then, for example, we may accept as true the above statement within the brackets. In support

1. Ayer, B.B.C. debate on Logical Positivism, included in Edward and Pap. (Ed) *Modern Introduction to Philosophy*, Pp. 586-618.

of his contention, he has mentioned the following statement: "This paper is white and not white all over" when the paper is really grey.

We have already given arguments against the use of such a statement as this. But the mistake in Ayer's argument is even more fundamental. He thinks that the law of contradiction depends for its truth on the conventional use of "not". Now, we certainly can change the usage of "not" as we like but this obviously can hardly affect the law at all. At best, we may say about a paper that is grey in colour that it is white and black all over, but not that it is white and not-white all over. Basically, the law of contradiction is so foundational that it is inevitable for any conceptual scheme concerning the world at any level whatsoever, perhaps even there where the language is just in its infancy, or even before it.

The other objection to the law of contradiction is that of Prof. Daya.[1] He has tried to show in one of his articles how a particular space can be called both red and not-red at the same time. In most of this article, he has tried to show the inability of the formal scheme

$$(x)\ (Fx.\sim Fx)$$

to designate or characterize any experiential or empirical situation whatsoever. Few, perhaps, would disagree with this. In the rest of the article, he has tried to show that the statement "the space p is both red and not-red" is a correct statement. Briefly, his argument is as follows:

> "Any finite space may be characterised by two different colours as, for example, a table may be red at one point and yellow at another. Similarly, the same space can be characterised by the presence or absence of the same colour at different times, as,

1. Daya Krishna—Law of Contradiction and Empirical Reality, Mind. Vol. LXVI, April 1957.

for example, a table may be red at one moment of time and yellow at some other moment of time."

This objection seems, in my opinion, completely unnecessary for no one really differs at all on this point. The statement concerning colours is always something like the following:

"If the table M_1 is red all over at time $t_1 \ldots t_n$, then none of its parts can be green at time $t_1 \ldots t_n$."

But as we have argued previously, if it be considered possible that two different colours can characterise simultaneously the same space (a contention that is outside Daya's objection made above), then this would be a reason for giving up the supposedly analytic statement concerning colours and not the law of contradiction, as Daya seems to think. The objection, at this point, that it is impossible to determine empirically the interval between the two moments $t_1 \ldots t_n$ for $t_n \ldots t_n + 1$ may not be amenable to measurement and thus the table may be not red at that time, is, in our opinion, not quite relevant. It is to make a linguistic question unnecessarily experimental. Perhaps the experiment may be relevant only to the extent if we ask ourselves whether anyone can be induced to make or assent to a self-contradictory statement in the situation. It is not relevant, in our opinion, to raise geometrical questions concerning "points" and "instants" in this connection, as Daya has made.

Daya presents another statement for consideration in this regard. According to him, we feel many a time that the proper answer to the question "Are you happy?" is "I am and I am not."[1] This usage is certainly accepted in the language also. But this, in my opinion, is not a self-contradictory statement and thus

1. Daya—"Religious Experience, Language and Truth," in *Religious Experience and Truth (Ed.)* Sidney Hook, New York University Press, 1961.

hardly raises any problems at all. We would, for example, not ask the person "How can you be both happy and unhappy at the same time?" But rather "what is the reason which makes you happy and what makes you unhappy?" Daya perhaps thinks that the above statement is of the same sort as the statement: "A table cannot be both square and round." Yet, it seems true, and therefore, it is an example of a statement which is both true and self-contradictory. But, he forgets, for example, that we cannot significantly ask in respect of a table as to "why it is round and why square?" It is, of course, not necessary in respect of a person that he be both happy and unhappy because of *different* things. It can be the same thing which may be the cause of both happiness and unhappiness, but then at least its aspects must be different. One may say, for example, "I am both happy and unhappy at being deceived by this friend of mine, because (1) 'being deceived by a friend is naturally painful' but (2) the reason for being glad is that I have found out soon what sort of a man he is." Here self-contradiction would occur only if it be said that "I am both happy and unhappy because my friend has deceived me, for to be deceived by a friend is both pleasant and unpleasant." But no one really speaks like that.

The issue of self-contradiction arises in the context of a conceptual scheme on the one hand and language on the other. In fact, it may safely be said that it arises *only* in these fields and in no other. But when a self-contradictory sentence is used, it is supposed to be empty till the contradiction is removed. The contradiction has no correspondence in facts. If a table is, then it *is*, the question of its not being there does not arise at all. Neither, for that matter, does the question of its both being and not being at the same time, or of the denial of such a conjunction arise also.

Of course, it is true that we do use on many occasions sentences which are full of self-contradiction.

For example, a liar may say "I saw Ramesh's friend this morning," and later in the course of conversation might say something which is directly or implicitly in contradiction with it. In such a situation we, obviously, would say that these two statements of his are self-contradictory. In the same way, a conceptual scheme can also be self-contradictory if there is a contradiction anywhere among the sentences formulating it. But even in these cases nobody can actually think or feel contradictorily. The fact that self-contradictory statements cannot have any content whatsoever relates them to the concept of truth. In fact, all the three types of statements, whether formal, analytic, or synthetic, are regarded equally as propositions because they are all empty when self-contradictory. If some one were to say, for example, that he saw table M_1 wholly inside the room at the time t_1 and asserts again that also at the same time t_1 he saw the same table partly outside the room also, then he certainly is uttering a self-contradictory statement. I hope Daya will agree with this also. Now, if such a statement is properly self-contradictory here, then such a type of statement should be considered self-contradictory everywhere. If some statement in some discourse appears self-contradictory in form but does not stultify itself, by that very fact, then, it is only *apparently* self-contradictory and not really so.

All this is very clear and yet Daya has claimed that self-contradictory statements are meaningfully and truthfully used in poetic and religious language. He claims that even if there be self-contradiction in the statements in a religious language, it is not a sign of their being empty of all cognitive content whatsoever. The example he adduced in this connection ("I am both happy and unhappy") we have already examined before. It has not been possible for us to understand how Daya could even conceive it to be possible. The argument of Prof. Ayer seems equally invalid in this connection. Suppose, we take the following statement:

"The ultimate reality is both finite and infinite, qualified and non-qualified together."

Now though there seems a contradiction here, it is only apparently so, in our opinion. Or you might say that in the context of a religious conceptual scheme such sentences are *not* to be understood as contradicting each other. Such statements are certainly often used in a religious discourse, but it should be equally clear that they do not assert the existence or non-existence of what is predicated. In fact, religious persons themselves say that they use such statements merely to convey the inability of all language to articulate or communicate the experience or the reality they wish to intend or express. Also, they say that the ultimate reality they are talking about is beyond both existence and non-existence, being and non-being. What these statements, therefore, imply is the fact that these sentences should not be understood as real contradictories of each other nor should they be construed as such. In fairness to them, it will be more correct to say that, bewildered at the insufficiency of language to convey what they wish to convey, they tend to use the sentences in such a way.

As we have already said, contradiction is primarily a relation between propositions and, thus, has no reference to any state of affairs that actually obtains. The same is true of analytic statements. But, many a time, we seem to deny the contradictory statements and assert the analytic ones in such a way as if we were saying something about the state of affairs obtaining in *rerum natura*. One may think, for example, "How could it ever be that Rama both ate his food in the morning and did not do so also?" Of course, it cannot happen like this in actuality, but this is not to say that there is any such legislation about it that it ought not to be so. The law of contradiction is not a law concerning states of affairs. But, perhaps, Daya does take it to be like that.

Equally, an analytic statement does not say anything about a state of affairs even if what it says may actually be true as a matter of fact. For example, the statement "All material things are extended" does not make a real statement about any state of affairs, even though it is true that material things are always extended in their nature also. If, on the other hand, the above statement is taken as stating the fact that all material things are extended, then it certainly would become synthetic in character in the same way as the sentences "All men are featherless biped" or "All birds are egg-born" are synthetic in nature.

However, the actual state of affairs may become relevant to both contradictory and analytic statements in another way. In the case of straight definitions, the state of affairs is, of course, completely irrelevant. The sentence "All bachelors are unmarried" will, for example, be true whatever the state of affairs. It can become false only by changing the meaning of the terms 'bachelor' and 'unmarried' themselves. But one may legitimately doubt both the truth and the analyticity of a statement such as "All material objects are extended." We had seen this before in the context of the doubts raised concerning the analyticity of the statement "Nothing can be both red and green all over at the same time." This is because implicit definitional statements are built not so much out of the terms contained therein as out of a specific selection from the "group of meanings" connoted by them, a selection which *alone* is designated as essential and necessary. Such statements are, then, doubted only on grounds of convenience or conceivability. The statement concerning the extendedness of material objects, for example, may be doubted on grounds of 'conceivability of the opposite' while the statement concerning colours may be doubted on grounds of 'convenience.' This can be done because the actual state of affairs seems to leave us free between accepting and non-accepting the analyticity of the statement

concerned.

Daya seems to raise the question of "convenience" even with respect to statements which are self-contradictory as, according to him, there are certain states of affairs which may adequately be described only by such contradictory statements. In such situations, therefore, contradictory statements function as descriptive statements. However, in our opinion, the question of "convenience" is unwarranted in the case of contradictory statements. Only, the question of "the conceivability of the opposite" is relevant and, as we have already seen, it is inconceivable that contradictory statements designate any state of affairs.

To sum up, primarily language is used as an instrument—to describe, to state etc., and as a describing activity it consists only of synthetic sentences. But the world it succeeds in describing is the world which is conceptually woven—a *meant* world. Now, this fabric of concepts or meanings forms an analytic scheme. Therefore, even when we use a synthetic sentence, e.g. "There is a table in the next room," we use it in the analytic fabric of meanings. Or else, how can we verify this sentence until we have a definite conceptual scheme of physical objects? Therefore, the sentences which bring out these implicit meanings, i.e., meanings presupposed by synthetic sentences, are analytic sentences. In fact, this is the realm where analyticity has an interesting role.

P.S.: This paper was written in the beginning of 1963 and subsequently I have slightly changed my views on the subject. For my present views see, ***Jnana Aur Sat*****. Rajkamal, Delhi, 1967.**

Frederick Suppe

SOME PHILOSOPHICAL IMPLICATIONS OF COMPLETENESS AND INCOMPLETENESS THEOREMS

In this paper some philosophical implications of the Godel Completeness and Incompleteness Theorems for first-order logics will be considered.

A *first-order theory* is a first-order predicate calculus augmented by a possibly-void set of additional axioms. Taking the form of the *Godel Completeness Theorem* which says if a first-order theory is consistent then it has a model, we obtain trivially as a corollary the following version of the *Lowenheim-Skolem Theorem :*

For any cardinal number $\alpha \geqq \chi_0$, any consistent first-order theory T has a model of cardinality α.

It is thus a fact about first-order theories that every such theory has model of each cardinality χ_0, χ_1,...... This yields some rather paradoxical results. For, if we assume the standard interpretation of the sentential connectives, quantifiers and predicates, but allow the remainder of the interpretation to vary, it follows that not only does the Peano axiomatization of the real number system possess its standard model of cardinality *c* (the real number system), but it also has a model of cardinality χ_0; similarly an axiomatization of the natural numbers has as well as its intended model (the natural numbers) of cardinality χ_0, a model of cardinality *c*. This result is sometimes known as *Skolem's Paradox* or the *Paradox of Lowenheim.* But as will be argued below, it is a rather unusual paradox as it embodies in it an essential defining property of the first-order theories in general.

The philosophically significant point to be drawn from this paradox is that, *within* first-order theories at least, it is impossible to axiomatize any theory to within uniqueness, for any attempt at first-order axiomatization of any system of necessity will admit of an infinity of different interpretations or models. Differently put: no structure or system can be uniquely characterized within a first-order theory. The unique specification depends upon the interpretation or semantics as well, and, as will be argued below, the specification of the semantics is of a higher-order and hence cannot be given within a consistent first-order axiomatization.

What is equally interesting is that essentially the same result also follows for a slightly smaller class of theories from *Godel's First Incompleteness Theorem* (general form, as presented in his 1934 lectures). The incompleteness theorem says that any first-order theory containing at least an emasculated and rudimentary arithmetic, if consistent, will be syntactically incomplete, that is, there will be an undecidable sentence **u** in the theory such that neither **u** nor its negation will be a theorem of that theory.

The proof of this theorem consists of two main steps: (1) a representation theorem is proved which says that all numerical attributes and properties can be represented in any theory of this class. Once this result is obtained for a theory, T, that theory must be of such complexity that self-reference is immediately allowed without loss of consistency. This self-reference is then exploited to obtain the second part of the result: (2) by an elaborate construction an undecidable sentence **u** which essentially says, "I am not provable," is obtained. It is then shown that **u** has the property that neither it nor its negation is provable in T. Hence T is syntactically incomplete. It immediately follows from the existence of **u** for any such T, that T admits of at least two different models of a given cardinality: one in which **u** is true and one in which **u** is false.

Here again the same philosophical consequence can be drawn as before: It is impossible to axiomatize a system to within uniqueness. Actually, our result is in one sense stronger than the one obtained previously. For there we knew only that there existed alternative models of different cardinality for a theory **T**, but the question was open whether there must be alternative models of the same cardinality. Here we have a partial answer to the question: If a representation theorem is forthcoming for **T** (i.e. if it contains the proper rudiments of arithmetic), then there must be alternative models of the same cardinality for **T**. Incidentally, it should be noted that this result cannot be avoided by electing to make **u** or **∼ u** an axiom of the theory **T**. For a consequence of the Godel result is that the new theory **T** so obtained must possess a new undecidable sentence **u**. The result will again follow.

We thus have the result, obtained in two different manners from Godel's Completeness and Incompleteness Theorems for first-order theories, that in axiomatizing a system, the system can never be axiomatized to within uniqueness within the theory itself. The unique characterization will depend as well upon the interpretation, model, or semantics of the theory.

Ideally we would like some explanation of this result, some reason why we cannot specify the interpretation within the language, thereby axiomatizing to within uniqueness. The answer is to be found in *Godel's Second Incompleteness Theorem*, also known as *Tarski's Truth Theorem :*

A formal language if consistent cannot define its own truth. Differently put: the definition of truth for a theory must be of a higher order than the theory itself.

The significance of this result for explaining our result concerning non-unique axiomatizability lies in the fact that in giving an interpretation to a theory **T** we

are defining truth for **T**. Now the second Godel incompleteness result tells us that if **T** is consistent such a definition of truth cannot be given within **T**. It is interesting to note that Godel has maintained in unpublished correspondence that this is perhaps the most fundamental property of formal languages and the true reason for the existence of undecidable sentences in formal systems containing arithmetic. That is, it is a fundamental feature of such formal systems or theories that "...a complete epistemological description of a language A cannot be given in the same language A because the concept of truth of sentences of A cannot be defined in A".[1]

It is precisely because of this that no first-order axiomatization of a system can be made to within uniqueness. Differently put: If the concept of truth for **T** could be defined *within* **T** then unique axiomatization would be possible; since truth cannot be so defined unique axiomatization is impossible *within* **T**.

From these two incompleteness theorems another rather surprising result can be drawn. It will be recalled that the first step in obtaining the first incompleteness theorem was proving a representation theorem—namely, that all numerical properties and attributes can be expressed in any such theory **T**. With an adequate Godel-numbering, anything that can be said precisely in any language can be represented or "said" in **T**. This follows primarily from the fact that any written language is essentially an arithmetic modulo-*n*, where *n* is the number of primitive symbols in the alphabet. Consequently, all sentences of a language can be mapped into equivalent statements about numerical properties of sets of numbers. Hence, whatever can be said precisely can be said in a

1. K. Godel, in a letter to A.W. Burks dated November 7, 1961, to be published in: J. von Neumann, *Theory of Self-Reproducing Automata*, ed. A.W. Burks, (forthcoming) University of Illinois Press.

first-order theory. But there are certain things which can be defined only on the pain of inconsistency—namely, the truth of the theory (as shown by Godel's Second Incompleteness Theorem). Von Neumann has put a few twists on this result, resulting in the claim that whereas the description of a system is generally less complex than the system itself, when the system exceeds a certain complexity the situation is reversed; the description of the system becomes of a higher order than the system itself.[1] And this level of complexity is precisely the level of complexity required to obtain the representation theorem which constitutes part (1) of Godel's proof of his first incompleteness theorem. This result is of philosophical importance because it tells us something about the complexity of truth.

In conjunction with the other philosophical conclusions we have drawn, an important consequence for the philosophy of science follows. For the remainder of this discussion we will restrict our attention to axiomatic scientific theories expressed in formal languages. The fact that axiomatizations of a system cannot be unique within a first-order formal language, the fact that uniqueness comes only by specifying the interpretation, that for complex systems the specification of the interpretation is of necessity of a higher level of complexity than the system being characterized, and the fact that whatever can be said precisely can be said in such a formal language, together suggest the following about axiomatic scientific theorizing within formal languages. Axiomatic science can characterize and theorize anything about reality that can be said precisely. But it cannot hope to have one unique formal axiomatic theory of

1. J. von Neumann, "Theory and Organisation of Complicated Automata" forthcoming in von Neumann *op cit.* Incidentally, this level of complexity is apparantly precisely the level of complexity required for self-reproduction.

science which covers all facets of reality for all purposes. For to do so would require specifying the interpretation of the formal language in which the theory is expressed within some language—formal or informal—which is more complex. But since the universal theory is as complex as reality any language in which the interpretation could be given would have to be more complex—hence more complex than reality itself. This is impossible since any possible language would itself be a part of reality and hence could not be more complex than reality. Consequently, axiomatic science must content itself with theories of limited scope, where the systems being characterized are of lesser complexity than reality, the scope of theories being limited by the limitations of language. Differently put: exact or axiomatic science can say everything to be said, but not at the same time.[1]

And there is one thing more that axiomatic science cannot do. To see this we now turn to a generalization of Godel's First Incompleteness Theorem, known variously as *Church's theorem on the decision problem* or *Turing's theorem on the halting problem.* For our purposes it will be simpler to look at Turing's proof, even though his approach is less obviously a generalization of the Godel result than is Church's. A formal system for which the representation theorem (1) above is forthcoming is equivalent to a universal Turing machine which can imitate the behaviour of any possible machine. The class of all machines can be bifurcated into two types; those which halt or stop computing after a finite time and those which do not halt continuing to compute for ever. Turing raises the question whether we can programme the universal Turing machine to separate the halting and the non-halting machines. Turing showed this to be impossible; from this, the fact that the halting predicate can be represented within such theories, and the

1. The similarity of this to the Heisenberg's indeterminacy result in physics is striking, intriguing, and perhaps worth investigating.

equivalence of universal Turing machines and formal systems or theories possessing the representation theorem, it immediately follows that there exists no decision procedure for such theories.

The philosophical point that follows from this result (aided by a few twists here and there) is that whereas we saw before that whatever can be said precisely can be represented in a first-order theory, now we see that we cannot predict within the theory what can be represented in that theory. This tells us one additional restriction on axiomatic science. We have already seen that axiomatic science must limit the scope of its theories in a way regulated by the inherent limitations of language. But within a theory also it cannot say what is and what is not within the scope of that theory. And this is reasonable, for to do so is equivalent to specifying the truth conditions of the theory, which we have seen cannot be done within the theory itself.

In summary, then, we have argued that the Godel theorems, together with some of their corollaries and extensions, aided by a few twists here and there, tell us something about the inherent limitations of exact or axiomatic science: any facet of reality that can be described is fair game for the exact sciences but every facet cannot be theorized about at once. Science is restricted to theories of limited scope, and no theory of any complexity can predict the limits of its own scope.

The writer would like to suggest that philosophy and philosophical theorizing is probably limited in precisely the same way as science; but this needs to be argued for.

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S. Bhattacharya

KNOWING THAT ONE KNOWS

PROFESSOR Jaakko Hintikka in his *Knowledge and Belief—An Introduction to the Logic of the Two Notions*[1] has developed a logic of knowledge by which he proves that it is indefensible to hold at once that *a* knows that p and that *a* does not know that he knows that p. We shall try to show here by an examination of the basic concepts of his logic that his proof rests on an assumption which he has not justified.

We first note the limitations within which Hintikka develops his logic of knowledge and belief (pp. 7ff). Some are analogous to those of ordinary formal logic. Just as we are told that the laws of identity and contradiction cannot apply to objects of this world of becoming, so also the logic of knowledge has to assume that the knowledge of a person neither decreases nor increases. As Hintikka says,

> "The statements in question must be made on *one and the same occasion...............*
> (i) the notion of forgetting is not applicable within the limits of an occasion,
> (ii) there cannot be any question of increasing one's factual knowledge except, perhaps, by following the logical implications of what one already knows or believes".[2]

1. Cornell University Press, 1962.
2. Ibid. p. 7

The most important condition which distinguishes Hintikka's logic of knowledge and belief is to be found in the clause

> "except by following the logical implications of what one already knows or believes."

Now it is obvious that this exception renders the laws of identity and contradiction inapplicable in this logic; we cannot say, for example, that if *a* does not know that p, then he does not know that p. For even if *a* does not know that p *now*, he may yet increase his knowledge by drawing conclusions from what he does know and *may come to know* that p. To avoid this difficulty Hintikka distinguishes between "actual or active knowledge' (p. 34) and 'implicit knowledge', and replaces the notion of 'inconsistency' by that of 'indefensibility'. Just as 'implication (equivalence)' is defined in terms of 'inconsistency', so also Hintikka defines 'virtual implication (equivalence)' in terms of 'indefensibility'. Hintikka's logic of knowledge is a logic of active and implicit knowledge. Let us see why he *has* to use the term 'knowledge' in the extraordinary sense of 'implicit knowledge.'

If we use the term 'know' in its ordinary sense (we are discussing only 'knowing that', ignoring the various other types of acts also called 'knowing' in ordinary language), then the number of propositions that a person knows is necessarily finite. If we symbolise 'X knows that p' by 'Wp', and call formulae beginning with 'W' 'W-formulae', then using 'knowledge' in its ordinary sense we can have only a finite number of W-formulae. If we want to develop a logic within this finite universe of discourse the logic is bound to be trivial and uninteresting. We show here why it cannot be even as strong as the basic modal logic (BML) in the sense of Lukasiewicz. Using 'knowledge' in its ordinary sense we have no difficulty in rejecting that X is omniscient, that is, we easily have the following axiomatically rejected formula:

A1. CpMp.

We may also assume easily that X knows at least one proposition, so that we reject the following

A2. NWp.

We may also agree that if X knows that p, then p must be true. So we have the axiom

A3. CWpp.

Now if we could only get

A4. EWpWNNp

the logic of knowledge would have been at least as strong as BML. (With 'X' defined as 'NWN', we can easily have the corresponding rejected formulae and axioms with 'X'.) But the difficulty is that we *cannot* have A4 as an axiom or as a theorem if we do not change the ordinary meaning of 'know'. For A4 generates an *infinite* number of W-formulae. From A2 we can assert that X knows at least one proposition, say p, and then we could say, if A4 were valid in this logic, that he knows an infinite number of propositions, e.g. NNp, NNNNp, and so on *ad infinitum*. For the same reason, logical laws like CCpqCWpWq, CKWpWqWKpq, cannot hold good in this logic. Yet without at least one of these laws no sufficiently strong logic can be developed. A logic based on A1-3, of course, can be regarded as (is, indeed, stronger than) a modal logic "in the more liberal sense of 'modal' " of Prior[1] (PML), but this only shows that PML can have a finite modal (and, hence, is not always interesting) whereas BML cannot.

To solve this difficulty Hintikka has followed the usual method of postulating 'limited omniscience', or 'a logical fiction, the rational man' who *implicitly knows* all the logical consequences of what he *actively knows*. Thus the assumption of a type of implicit or non-active knowledge which will expand the set of W-formulae into

1. A.N. Prior, *Time and Modality* (Oxford University Press, 1957).

an infinite set seems necessary. (This corresponds to the metaphysical theory that through reason man transcends his finitude.) Hintikka thinks that this assumption of rationality is relevant for his theorem that *a* knows that p virtually implies that he knows that he knows that p. He says:

> "In his contribution to the symposium "*Is there only one correct system of Modal Logic*" (Proceedings of the Aristotelian Society. Sup. Vol. XXXIII, 1959, 23-40) E.J. Lemmon rejects (p. 39) the implication from (64) to (63). He rejects it in spite of the fact that he is not concerned with active knowledge but only with "a kind of logical fiction, the rational man" who (implicitly) knows all the consequences of what he knows. Hence Lemmon seems to be concerned, in effect, with the same notions as we; he seems to reject the virtual implication from (64) to (63). This would be rather serious for our purposes, for rejecting this virtual implication would necessitate rejecting the condition (C.KK*) on which the proof of the implication rests. Lemmon's reasons are not, however, valid against what I have said. They are in terms of what "the rational man" might (rationally?) forget. They are therefore ruled out by the initial provision that only statements made on one and the same occasion are considered here."[1]

Here we shall try to show that this theorem which, as he himself says, is based on "(CKK*) or, if you prefer, (A.PKK*)" (p. 110), has got nothing to do with 'the rational man' or 'forgetting' (although in Lemmon's argument 'forgetting' is used), but follows from *another assumption* on which both (A. PKK*) and C.KK*) rest and which Hintikka has not stated, let alone justified. We now study these conditions in detail.

1. *Knowledge and Belief.* Footnote Pp. 105-106.

II

Explaining the justification of (A. PKK*) Hintikka says:

"Let us suppose some one makes a number of statements on one and the same occasion; and let us suppose that among the sentences he utters there are the following:

" I know that p_1. "
" I know that p_2. "
......
"I know that p_k."

"It is possible, for all that I know, that q."

Under what conditions is he consistent? It will be agreed that one condition at least has to be fulfilled, namely, that the following set of sentences is also consistent:

" I know that p_1. "
" I know that p_2. "
......
" I know that p_k. "
"q is in fact the case".

In other words: If it is consistent of me to say that it is possible, for all that I know, that q is the case, then it must be possible for q to turn out to be the case without invalidating any of my claims to knowledge; that is, there must not be anything inconsistent about a state of affairs in which q is true and in which I know what I say I know.

In formal terms, we may formulate the condition in question as follows:

(A.PKK*) If a set λ of sentences is consistent and if "K_aP_1" $\varepsilon\lambda$,
"K_ap_2" $\varepsilon\lambda$, ..., "Kap_k" $\varepsilon\lambda$, "Paq" $\varepsilon\lambda$,
then the set
{"K_ap_1", "K_ap_2",..............., "K_ap_k", q}
is also consistent.[1]

(Hintikka uses "Kap" for "a knows that p" Pap" for, "it is possible, for all that a knows, that p".)

1. *Knowledge and Belief*, Pp. 16-17.

Elaborating further he says:

"The rule (A.PKK*) may be compared with the weaker rule—we shall call it (A.PK*)—obtained by replacing the set {"K_ap_1", "K_ap_2",......"Kap_k", q } by {p_1, p_2 ......p_k, q}. Why are we justified (as we clearly are) in adopting the stronger rule (A.PKK*) and not the weaker rule (A.PK*)? One answer to this question seems to be as follows:

That q is the case can be compatible with everything a certain person——let us assume that he is referred to by *a*——knows only if it cannot be used as an argument to overthrow any true statement of the form "*a* knows that *p*." Now this statement can be criticized in two ways. One may either try to show that *p* is not in fact true or else try to show that the person referred to by *a* is not in a position or condition to know that it is true. In order to be compatible with everything he knows q therefore has to be compatible not only with every *p* which is known to him but also with the truth of all the true statements of the form "*a* knows that *p*." And this is exactly what is required by (A.PKK*)[1]."

Hintikka justifies his (C.KK*) in the same way. He says:

"The condition (C.P*) serves to make sure that it is possible that *p*. We required more, however; we required that it is possible, for all that the person referred to by the term *a* knows, that *p*. Hence everything he knows in the state of affairs described by μ he also has to know in the alternative state of affairs described by μ^*. In other words, the following condition has to be imposed on the model sets of a given model system:

(C.KK*) If "K_aq" ε μ and if μ^* is an alternative to μ (with respect to *a*) in some model system, then "K_aq ε μ^*"[2].

And he also accepts (C.K.) If "Kap" $\varepsilon\mu$, then $p\varepsilon\mu$. We may ask here whether this analysis is correct.

1. *Ibid.* Pp. 17-18.
2. *Ibid.* p. 43.

Suppose there are two persons X and Y, and ask X the following questions:

Is p_1 true? Is p_2 true? ... Is p_k true? Is q consistent with p_1, p_2......p_k?

Suppose X answers:

Yes, p_1 is true, for......, p_2 is true, for...........; ....p_k is true, for......; q is consistent with p_1, p_2......p_k, for....

Now suppose Y reports as follows:

"X knows that p_1."
"X knows that p_2."
"X knows that p_k."
"It is possible, for all that X knows, that q."

We cannot ask here, like Hintikka, "under what conditions is he consistent?" For now there are two persons and we have two different questions:

(i) Under what conditions is X consistent?
(ii) Under what conditions is Y consistent?

The difference may be explained as follows:

Suppose X says:

"the table in that room is brown",
and Y reports:
"X knows that the table in that room is brown."

Now this statement of Y can be 'criticised in two ways':

Firstly, by showing that the table is, in fact, not brown which will also prove the falsity of what X has said:

Secondly, by showing, for example, that X is colour-blind, or that he is dreaming, even though the table is actually brown.

But this method, unlike the first method, does *not* refute what X has said, but only the statement of Y. This shows that X and Y have not made the same claim. Now suppose X says:

"the table in that room is brown for it belongs to a person, who owns only brown tables"

and Y reports:

"X knows that the table in that room is brown."

(If Y had reported "X knows that the table in that room is brown for etc.," then we have got the first case over again.) Can this statement of Y be 'criticised in two ways'? Can we now say that X is 'not in a position or condition to know' that the sentence 'the table in that room is brown' is true? The only method to criticise Y is to show that the table is not brown or that it does not belong to one who possesses only brown tables: that is, we can refute Y only by refuting X. Does this mean that the standpoints of X and Y have ceased to be different, that they have made the same claim? The answer to this question depends on the answer to a different question. Why the other method of refuting Y fails here becomes clear if we note that what X says *justifies* Y in reporting that X knows what Y says he knows. So any question about X's not being in a position to know it cannot be raised at all. But does this mean that X knows that he knows (although Y *has* known that X knows)? The answer to this question, on which the answer to the previous question depends, has to be negative. When X says that the table in that room is brown for it belongs to one who owns only brown tables, he says something about objects, but nothing about *himself*. Reasoning *about objects* is neither introspection nor another way of knowing mental states. But then we may ask:

How can Y know about X's knowledge when X himself does not?

To answer this question we must distinguish between what an inference proves and what the *fact* that the inference has been made by someone proves. X makes the inference and knows only objects. But Y makes *another* inference, that X knows for he has made a valid inference. X himself can know this if he makes this second inference, but his first inference cannot yield the knowledge which the second yields. From *what* X knows, there is no logical inference to *that* X knows, or to the *existence* of X's knowledge. The inference of the existence of X's knowledge is always from the existence or actuality of a certain type of behaviour of X. Existence can be logically inferred only from existence. Thus there is nothing wrong in supposing that something may be consistent with *what* X knows without being consistent with the fact *that* he knows. The standpoints of X and Y remain different even if they happen to be one and the same man. This is what Professor Ryle achieves by his theory of orders of knowledge. When X makes the first inference he has a first order knowledge; when X* makes the second inference he has second order knowledge (i.e. knowledge about knowledge). The way in which X knows his mental states is the same as the way in which Y knows them. There is no 'privileged access' to one's own mental states. The fact that I am *rational* does not help here at all. For knowledge of my own mental states is empirical knowledge which cannot be logically derived from what I know, but is based on various relevant observations. Hintikka's justification of (A. PKK*) and (C.KK*) rests on an identification of the standpoints of X and Y for which he has given no reason at all. Thus Hintikka's claim to "have restored one of the conclusions Ryle rejected, namely, that knowing something virtually implies knowing that one knows" (p. 111) is not justified.

Thus the real nature of Hintikka's logic of knowledge becomes clear. Even if we agree with him that the logic of knowledge deals with sentences of the form 'X knows that p', still we have to distinguish here between the standpoints of X and Y. It is obvious that the conditions under which X is consistent are precisely the conditions which Hintikka states in his rejected rule (A.PK*), and (C.K.). His (A.PKK*) and (C.KK*) state

the conditions under which Y is consistent; that is, Hintikka has developed a logic of knowledge from the standpoint of Y. There is nothing wrong in this, but those who, like Ryle, make the distinction between the standpoints of X and Y, will point out that Hintikka's theorems do not, in general, remain valid when interpreted from the standpoint of X. Hintikka has not given any reason to show why the distinction between the two standpoints cannot be made, and has failed to note that he and Ryle are talking of entirely different things.

III

Before we conclude we note another feature of the logic of knowledge which Hintikka has developed on the basis of (A.PKK*). There is a sense in which this logic is trivial, for it is only an interpretation of the ordinary (i.e. non-epistemic) modal system S4 of Lewis. Hintikka has, therefore, given no law which is peculiar to the logic of knowledge. We sketch here a method of developing a type of epistemic logic with special laws, which also allows us to get rid of 'the logical fiction, the rational man.'

Instead of trying to build up our systems on the basis of propositional calculus (PC), we take a system of modal logic as the basis. If this base is not merely an alethic modal logic but is also a deontic logic, then various other concepts can be defined and explained. We follow here this procedure and take S4.2 as the base which, as has been shown by Dawson,[1] can also serve as 'a normal deontic logic' in the sense of Anderson. In this system 'P' standing for 'it is permitted that' is defined as 'LM', and 'O' standing for 'it is obligatory that' is defined as 'ML'. Now we give the axioms.

System SKMI

Base—S4.2.

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1. *A Model for Deontic Logic. Analysis* 1964.

Axioms:

B1. CMWpp.

B2. CpMWp.

B1 and B2, we hope, do not change the ordinary meaning of 'know'. For according to the ordinary usage a false proposition cannot be known (CNpNMWp) which is what B1 says. B2 is the axiom which makes the number of available propositions infinite without making X omniscient or changing the meaning of 'know'. For, we may agree that although X is not omniscient, yet it is possible for him to increase his knowledge. B2 asserts there is no unknowable true proposition. We have replaced Hintikka's 'implicit knowledge' with 'possibility of knowledge.' Hintikka assumes that a rational man implicitly knows all the consequnces of what he actively knows, whereas B2 asserts that it is possible for X to know any true proposition whether it follows from what he already knows or not. We can derive easily A3 from B1 with the help of the modal law CWpMWp and hypothetical syllogism. But we cannot get controversial laws like CWCpqCWpWq. As Lemmon has pointed out against von Wright, "X may very well know a conditional to be true, and know its antecedent to be true, without drawing the conclusion, and so without knowing the consequent to be true."[1] Hintikka's reply here is that although X does not know actively the consequence, yet he implicitly knows it. But our theorem (which can be easily proved), CWCpqCWpMWq, only says that it is possible for X to know that q. So also with the other theorems. The logical peculiarity of SKM1 also follows from B1 and B2 which together imply the equivalence of p with MWp. There is no logic, truth-functional or modal, in which two different monadic functors combined in a certain order cancel each other. SKM1 can be developed on any modal system including

1. *Aristotelian Society Proceedings.* Supp. Vol. XXXIII, 1959, p. 38.

purely alethic ones, for its axioms B1 and B2 do not contain any deontic functor. But the following alternative systems depend essentially on a deontic base:

System SKM2

Base—S4.2.

Axioms:

B3. CMWpp.

B4. CpPWp.

System SKM3

Base—S4.2.

Axioms:

B5. CBWpp.

B6. CpMWp.

System SKM4

Base—S4.2.

Axioms:

B7. CPWpp.

B8. CpPWp.

System SKM2 is the strongest and SKM3 the weakest of these four systems, SKM2 contains two equivalences, EpMWp and EpPWp, from which we have EMWpPWp. That is, what is possible to know is also permissible to know. This seems to go against the 'normal deontic logic' in which Mp does not imply Pp. And this non-implication is quite justified, for otherwise everything that we actually do also becomes permissible, so that performance of an immoral act becomes impossible. But in the case of knowledge, we have already guaranteed against all 'errors of commission' by stipulating that we cannot know a false proposition. The only 'sin' that we can commit in knowing is, thus, 'an error of omission'. That is, we may fail to know what we ought to know; indeed, every finite being is finite (in knowing) precisely because of this reason. Hence in knowing there is no 'sin' corresponding to the sin of performing a prohibited act of ordinary deontic logic. Thus in the logic of knowledge we may equate the possibility of knowing with permissibility. In this case the equivalence is not counterintuitive.

In these systems we can define the concept of 'logically following' thus:

DB1. q logically follows from p $\overline{\overline{\text{df.}}}$ CWpOWq.

Here also we note our difference from Hintikka who does not appear to be consistent in his theory of logical laws. At one place he says:

> The fact that the so-called laws of logic are not "laws of thought" in the sense of natural laws seems to be generally admitted now-a-days. Yet the laws of logic are not laws of thought in the sense of commands, either, except perhaps laws of the sharpest possible thought. Given a number of premises, logic does not tell us what conclusions we ought to draw from them; it merely tells us what conclusions we may draw from them——if we wish and if we are clever enough........."It is not an inference," a justice has put it....... "that must be drawn, but which may be." This is the case, I submit, with all logical inferences (as such).[1]

Here Hintikka seems to say that "q logically follows from p" means "q may (not must) be inferred from p". In our terminology this will be equivalent to the following definition:

DB2. q logically follow from p $\overline{\overline{\text{df.}}}$ CWpPWq.

Now if an act is permissible without being obligatory, then the non-performance of it also is permissible, (CNOpPNp is a law of the usual types of deontic logic); and for not doing it we cannot be condemned or criticised in any way. So if the laws of logic only permit us to draw certain conclusions from certain premises, then no one can be criticised for not drawing them. But strangely enough Hintikka does not seem to accept this

1. *Ibid.* Pp. 37-38

position. For, he also says:

> There are no logical reasons why somebody who knows that p should know that q even when q's following from p is perfectly obvious. But such cases are likely to be exceptions. If the consequence is quite obvious, we might even be reluctant to say that he does not know that q, although he denies himself, on being asked, that he knows it,.........But even apart from questions of legal responsibility you do not willingly expose yourself to criticism to which you can only reply by admitting your failure (inability) to see the implications of what you are saying...[2]

Here Hintikka seems to mean that it is not merely possible, but obligatory on our part to draw the consequences which follow logically from what we know. For, by not drawing them, we 'expose ourselves to criticism' or have to 'admit failure or inability'. Thus Hintikka seems to waver between two different theories of the nature of logical laws——

(i) the laws of logic are necessary like the laws of ethics, so that even though we may not actually obey them, yet we are then criticised for failing to do our duty;

(ii) the laws of logic merely permit us to draw certain consequences from certain premises.

We now sum up. Systems SKM1-4, although presupposing a stronger base than Hintikka's logic, have the following advantages over it:

(1) these logics use 'know' in its ordinary sense;

(2) they do not need the postulate of rationality;

(3) they deal with sentences of the form 'X knows that p' from the standpoint of X;

(4) they show some special laws of knowledge which are not found in any other logic.

2. *Ibid.* Pp. 34-35

A.P. Rao

MODERN LOGIC AND ONTOLOGY*

(1)

IF the heart of philosophy is ontology, as the Iowa philosophers believe it to be, when x-rayed by modern logic it shows the signs of a misplaced heart. The burden of this note is to establish that the cardiac centre of philosophy is not at a point where Quine encountered the three-word puzzle, but at a point where the four-word conundrum 'how many are there' is confronted.

(2)

Let *Fanalytics* be the formalised version of the *Analytics*. The *Fanalytics* provides all the rubrics required for a treatment of functions with one argument, but not functions of higher degree. This makes it an inadequate framework for axiomatising classical mathematics. Whatever might be the historical reason, this one seems to be the genetic reason for the growth of logic during the last hundred years. This will he evident when statements about the whole universe of discourse, which can be found, not infrequently, in classical mathematics are taken into consideration. Let K be the

* A part of the paper originally presented at the Seminar was published in the *Visva-Bharati Journal of Philosophy*. Vol II, under the title *Formalisation and the Ontological Issue*. The present note constitutes the other part of it, slightly modified in order to make it a unit by itself.

class of all natural members. Let f be a function defined over the class K. To every function of one variable f(x), the class of all, and only, those entities e of K, which satisfy f(e), can be correlated. Let this class be M. Suppose that one-to-one correspondence between K and M is possible. Then the *Fanalytics* is devoid of any apparatus to deal with K and M. Further, within the framework of the *Fanalytics* it cannot be asked, meaningfully, whether or not the function f(x) is identical with zero, for in it classes without entities are illegitimate. But the theory of equations requires formulas like $f(x)=0$ and $f(x)\neq 0$. The implication is that the *Fanalytics* cannot comprehend functions whose arguments range over zero or infinite number of values. (No wonder if Aristotle *denied* the possibility of having true statements about the non-existent and the infinite entities.)

To make arrangements for a treatment of zero and infinity two new entities had to be brought into the universe of discourse of the *Fanalytics*, modifying it suitably keeping the requirement in view. This task was taken up and accomplished by Boole, the two new entities being the *null-entity* and the *universal-entity*. The null entity is one that if it satisfies a given function, then no other entity does satisfy that function, and the universal one is such that if it satisfies a given function every other entity does satisfy that function. To see a little more of these two unique entities, let K be the class defined over the identity function, and let f(x) be the negation of that function. To the function f(x) and to the class K, there corresponds a class M, whose elements are precisely those entities x of K which satisfy the function f(x). Now if e were to be an entity such that f(e) is true, by virtue of being an element of M as well as of K, e satisfies both f(x), and the function of which it is the negation. From this it follows that e is an element of every class (without contributing to its cardinality), and, on the other hand, it is not an element of any class (except the unit class of which it is the lone element) for it does not satisfy the function to which the class can be correlated. The acceptance of these entities does not involve any metaphysique, for all that is meant by an entity, here, is that which satisfies a function and nothing more. However, the usually posited or accepted relation between them, namely, that they are not identical, does require a metaphysique (in the broad sense of the term) for it tantamounts to asserting that the universe is not empty or that there *are* entities which are neither null nor universal.

And when something true is said about a class of entities, it comes out true with respect to every entity of that class except the null entity. Thus it eludes being talked about. This, however, is only one of the faces of Janus. To have a look at the other, allow f(x) to be a function satisfied by all cardinal numbers, and belonging to a language L which contains as a part a substantial part of arithmetic. Obviously L contains all the functions satisfied by these entities. Now consider the cardinal number of the class of these entities. It cannot be subsumed under the function f(x). This, just, implies that when all entities are talked about the universal entity slips away. In this case not all cardinal numbers, but all cardinal numbers of a given *type* can be talked about. (This is how Cantor's aleph series and Russell's theory of types came into vogue.) The import is that the unrestricted generality implicit in the assumption that f(x) is satisfied by the whole universe of discourse will have to be rejected.

Between these two elusive entities, namely the all-exclusive null entity and the all-inclusive universal entity, lie the tangible entities that can be caught in the propositional net.

(3)

As is obvious, the differences between the nominalists and the Platonists have been frequently reported to be

due to the former's stricture that the net should be used to catch *individuals*, and the Platonists' ambition to catch all kind of entities, be they individuals or not. Nothing, they believe, is too nimble for their net. But the real reason for the tribal war between them is not so much as to what can, or ought to, be caught, as much as it is with the quantity of the catch. When a nominalist notices that his catch is just a lone shrimp, he reconciles himself to his bad luck or poor ability and attempts again. In a similar situation, we are told, a Platonist would be happy at his haul, for he finds in his take besides the lonely shrimp, the class of this lone shrimp, the class these two, and so on. Both of them might (for a Platonist can hardly practise his philosophy) count only one acquatic animal in their respective nets. But while taking an ontological account of the haul, the nominalist counts one, and the Platonist finds the counting numbers insufficient for the census.

For a nominalist, an entity and the class in which it is the lone member are *numerically* one, and hence a count on one exhausts the other. And the all-inclusive entity which exhausts all other entities is not numerically distinct from them in the sense that a count on those entities exhausts the universal entity. And only in justifying this mode of counting he would come out with the idea which has been taken to be the cardinal thesis of nominalism, namely, that *abstract names are not names of abstract entities*. But what has crept into his argument by way of explanation has been taken to be the aim of the argument itself.

However, the nominalist tries to eschew any assumption as to the total count of the entities in the universe. This, at least, is his proclaimed intention, though in practice he quibbles a lot. And the Platonist has no scruples whatsoever as to this. But when noted with some advertency, it can be seen that the nominalist refrains from committing himself as to the number of entities that the universe *at most* can contain. A commitment as to the *least* number of entities that it will have to contain becomes imperative for everyone, whether he is a Platonist or a nominalist. For whatever might be the entities that the universe is supposed to be stacked with —be they classes or individuals—it will have to be accepted that it contains at least one entity. The theory of classes as well as the theory of individuals take only non-empty domains as models. In fact, a theory of empty domains is impossible as by definition a theory is an ordered n-tuple where the first term is a non-empty set of entities constituting the domain; the null element may occur as a term in this n-tuple (though a good nominalist, like Goodman, does not tolerate even that) but it itself cannot be the set constituting the domain.

As to the total number of entities in the universe, nominalists declare : "we decline to assume that there are infinitely many objects."[1] This ten-word declaration is defended by Quine and Goodman as follows : "....not only our experience is finite, but there is general agreement among the physicists that there are no more than finitely many objects in all space time. In fact, the concrete world is finite; acceptance of any theory that presumes infinity would require us to assume that in addition to the concrete objects finite in number there are also abstract objects."

The last of these sentences seems to create the impression that the admittance of infinity follows from the admittance of abstract entities. But even if we prefer to have a *cat*-with-grin and drive away the *grin*-without-cat, there are itchy issues. After all, a theory if, at all, has a model with the domain comprising of only a finite number of grinning cats, will also have a model with the domain comprising of denumerably infinite number of

1. Quine and Goodman : Steps towards constructive nominalism, *Journal of Symbolic Logic*, Vol. XII.

grinning cats; and if it has a model of cardinality aleph null then it has a model $\propto >$aleph null, unless it is an inconsistent theory. Thus, every consistent theory, as Skolem's peeling has shown, has a built-in apparatus for infinity. This implies that we can have a theory of $\propto \leq$ aleph null concrete entities. Hence it is not abstractness or concreteness of entities that is the determining factor in the nominalist-Platonist controversy.

(4)

Further, if the nominalists' rejection of infinity (or the acceptance of finitude) is to be based on the experienced magnitude of the universe, as is suggested in the first sentence of the passage quoted above, then they will have to first make it clear whether the entities are concrete because they are finite in number, or they are concrete because we encounter at any given moment of our mortal life only a handful of entities and entities only to the tune of that number will have to be bestowed concreteness? Moreover, a Platonist (at least a non-nominalist) can also extend an analogous argument. If the aim is to give an explanation of the experienced universe, we may as well accept a finite number of classes and bestow upon them concreteness. Not even the most dogmatic Platonist, I believe, would say that he encountered more than finite number of classes in his life. And now, if a nominalist and a Platonist differ, the differences would be due to the differences in their respective epistemological points of view. They would, then, be differing as to whether what we experience is composed of classes or individuals, and this difference does not belong to ontology *sensu stricto*, as it would only be an epistemological analysis of experience with a certain kind of ontology as a tool.

(5)

Then, what, precisely, is it that brings ontological differences between the Platonists and the nominalists? It is not, as is hinted already, the former's conceiving the universe as a crate of classes and the latter's as of individuals. Nor is it due to the former's commitment to infinity and the latter's to finitude. The difference, essentially, is due to the respective principles of enumeration. The nominalist's motto is: if from *n given entities* (whatever they might be—classes or individuals, abstract or concrete) and by a principle of construction, an entity can be constructed, the *constructed entity* is not numerically distinct from the given *n* entities. It is disregard for, and violation of, this principle that keeps the Platonists in the opposite camp. Thus, to put it succinctly, the nominalist holds that the universe constitutes as many, and no more or less, entities as are given in it, and the Platonist thinks that it has place for those *entities* that are given in it plus all those that he can construct from them or out of nothing. (This explains why nominalists suspect the epsilon operator, Russell class, descriptions etc. If the latter two, for him, are bereft of (independent) ontic status, the former is devoid of any constructive role.)

A nominalist, in order to sustain his thesis, need not assume either the infinitude or the finitude of entities (as, incidentally, Hao Wang[1] has thought). All that he needs to bar multiplication of entities, and to maintain that *there are as many entities as there are*—a tricky but not trivial idea—is to insist that every theory must contain as many *concrete names* (that is, names of the objects of the domains of interpretation) as there are entities in the domain of interpretation. (This at least will keep out cardinals higher than aleph null; and no higher cardinals are needed to formulate and develop a satisfactory theory of real numbers or theory of continuous functions.)[2]

Abstract names might occur in a theory; and a nominalist has no objection either to their occurrence or to their use. He will only say that they have no referents that share ontological fraternity with the referents of the

1. What is an individual? *Philosophical Review*, vol. LXII.
2. See Fitch, A Theory of Logical Essences, *Monist*, vol. LI.

concrete names. He will not even say that statements containing abstract names are meaningless—that is, neither true nor false. He would argue, of course, that it is not statements taken individually that are either true or false, but whole theories. Thus a statement containing an abstract name is meaningful in the context of its occurrence in a theory. And the truth or falsity of a theory containing abstract names, besides other things, is determined by the principle used in constructing abstract names from the concrete ones occurring in the theory. This principle bars multiplication of entities and the numerical distinctness of the purported referent of an abstract name from the referents of the concrete names that go into the construction of it.

(6)

With reference to theories incorporating some principle of construction of abstract names from concrete names, nominalists take suitable precautionary steps to preclude any violations of the principle that there are as many entities as there are; for instance, Quine's "principle of existence." Not that Quine came to hit at the equation of 'to be' and 'to be the value of a variable' for specifically these reasons, but that is, to use Bergmann's pet phrase, the structural relationship between Quine's principle and the thesis of nominalists. Quine's principle is to the effect that for any theory T, and for a statement S of T, S violates the nominalistic thesis, and thereby multiplies entities, if and only if it does not imply statements $S_1 \ldots\ldots\ldots, S_n$ of T or some truth-functional compound of these, where $S_1 \ldots\ldots\ldots, S_n$ contain quantifiers over variables, the substituends of which are only the concrete names of the theory T. This closes the doors of the ontic chamber for abstract entities, and keeps the number of "existent entities" in one-to-one relationship with concrete names.

Thus, what is central and decisive to ontology seems to be not what there is, but how many there are. Ontology is not qualitative but quantitative.

R.C. Pandeya

FUTURE CONTINGENTS

THE main problem that arises out of our consideration of future-tensed statements is not logical as many philosophers and logicians have taken it to be; to me this problem is a-logical and its solution may, therefore, be found in separating the logical from a-logical problems. Here an attempt has been made to show (i) that the view of those who think the main problem to be logical fails to solve it, and (ii) the direction in which the solution could be found. I have not been able to elaborate the method of the proposed solution here, but I am confident that this solution can be fruitfully worked out.

I

In the system of the two-valued logic, a declarative sentence must necessarily convey a proposition which is either true or false. But a statement about the future, made now, cannot be said to convey any proposition, true or false. Shall we then regard that (a) propositions conveyed by these future-tensed statements, though they have a truth-value, yet the sense in which they are held to possess a truth-value is different from that of other propositions which are about the past or the present? Or, (b) the kind of the truth-value of propositions conveyed by future-tensed statements is different from the two traditionally accepted truth-values? The first is an attempt to solve the problem within the two-valued

system of logic while the second view seeks the solution in a three-valued system.

Let us examine the problem as it poses itself within the two-valued system of logic. Let us start with examples:

It will rain tomorrow. (1)

Within the two-valued system, we may enquire whether the proposition is true *now*, that is, when the statement conveying this proposition is made. This question does not arise in the case of:

It is raining. (2)

or

It was raining. (3)

Because, if asked, it will not be relevant. The question whether (2) is true now is redundant and that whether (3) is true now is asking for the obvious. Thus (2) and (3) are true now because the fact they refer to is very much present there to provide a guarantee for their truth-value. The truth of a proposition and the fact in question are found together in the case of past-and present-tensed statements. People try to treat (1) also on a par with (2) and (3). Thus, if it is asked whether (1) is true now, the answer that these people may give would be (1) is true now or (1) is false now. If it is true now, then on the analogy of (2) and (3) there must necessarily be a corresponding fact. Hence, rain is bound to come tomorrow; nothing can prevent it, just as nothing can falsify if (2) or (3) is true. Similarly, if (1) is false now, it cannot be true subsequently.

Attempts have been made to separate time-considerations from the truth or falsity of a proposition. For example, (2) may be true but I may be prevented from looking through the window and also from hearing any sound coming from outside my room. In that case although (2) is true but it is not known by me to be true or false. Likewise, (1) must be true or false, when the statement is made, but we are unable to know it to be true or false. Just as (2) or (3) is necessarily either true or false (although one may not know its truth value) similarly (1) is also necessarily true or false, but its truth value remains unknown. The knowledge of the truth-value of a proposition will have, therefore, to be distinguished from its truth-value. The distinction between the *knowledge* of the truth-value of a proposition and the *truth-value* is relevant in the case of future-tensed statements because it explains our inability to answer questions like whether (1) is true now. The answer in that case would be: I do not know whether (1) is true now, rather I cannot know; but (1) is necessarily either true or false now, as ever. 'Now' in the question is not related to the truth but only to our knowledge of the truth. In the case of (2) and (3), 'now' can be related both to truth as well as to its knowledge because at least in principle it is possible to know whether a proposition in question is true.

But even if this distinction is accepted, a case for the acceptance of a kind of fatalism is still maintained. There is a logical necessity in $p \vee \sim p$. As shown above, it is assumed that a necessity similar to this also holds between $p = (1)$ and the rain next day in the case of (1), just as such a necessary relationship is found to exist in the case of (2) or (3). Thus if (1) like (2) or (3), is true, it is immaterial whether I know it to be so or not, the event in question is bound to take place, provided a two-valued system of logic is accepted.

II

Lukasiewicz sought to remove the difficulty concerning the truth-value of a statement about the future, by formulating a system of three-valued logic. In addition to the two traditional values of propositions, he introduced a third value—the possible. Future contingents are neither true nor false; they are just possible. Thus (1) would mean that rain is possible the next day. Consequently the value of (1) today would be 1/2 (i.e. the possible).

But this value, being intimately linked with time, will have to change according as the event takes or does not take place. Thus:

(i) [at t_1] $p = 1/2$

(ii) [at t_2] $p = 1$ or $p = 0$

A meteorologist, for example, says today (at t_1) "It will rain tomorrow". The next day (at t_2), I find it raining. I can then (at t_2) say that his statement (made at t_1) is true. On the basis of observation at t_2, the proposition which was only possible at t_1 becomes true at t_2. Lukasiewicz seems to think that a proposition about the future is intrinsically possible; our incapacity to know its truth-value does not make it possible. Had this not been the case, he would not have insisted upon the possible being the value of a proposition. In that case he would have maintained that although a proposition was either true or false yet my knowing it as true or as false was possible in time to come. It is also implied that just as in the case of (2) or (3) there is a necessary relation between the fact and the truth value, similarly there must also be such a relation between the fact and the truth-value of (1). But since the event in question is probable in this case, the truth-value of the proposition is also, correspondingly, possible.

It is necessary to make a distinction between the time when a statement is uttered and the time which is a part of the proposition conveyed by a tensed statement. This distinction, though obvious, is quite significant in this context. Let us see how this distinction holds in the case of (2) and (3). Thus when (2) is expanded, we get

[A says at t_1 that] it rains at t_1. (2a)

When (3) is expanded, we get

[A says at t_1 that] it rains at minus-t_1 (3a)

or

[A says at t_2 that] it rains at t_1.

In both these examples, the time indicated within the square brackets is the time when a statement is made and the time mentioned out of the brackets is the time which is a part of the proposition. In the case of (2) the time of making a statement and the time which is a part of the proposition are the same. In the case of (3) the time of making a statement is posterior to the time which is a part of the proposition. Thus (3) can be converted into (2) without changing its truth-value. So the uses of 'is' in (2) and 'was' in (3) do not make any difference so far as the time of the event in question is concerned. The time forming a part of a proposition remains unchanged, although expressions designating it may vary according to the rules of grammar. Tenses simply indicate the time when a statement is being made or when the truth of a proposition is being asserted. I will call the time expressed by tenses, that is, the time when a statement is made, as *k-t* and the time forming a part of the proposition as *p-t*.

The difference between (2) and (3) lies only in respect of k-t; so far as p-t is concerned they are inter-changeable as will be clear from comparing (2a) and (3a). Can we hold a similar position with regard to future-tensed statements also? The use of 'will' in (1), for example, is indicative of the time when rain is likely to occur. Following our method of expansion, we can say with regard to (1) that—

[A says at t_1 that] it rains at plus-t_1.
Or
[A says at t_1 that] it rains at t_2. (1a)

In this case, unlike (3), *k-t* is anterior to *p-t* and this fact is indicated by the use of 'will.' Advocates of the two-valued logic will say that, in the present case, nothing can be said about *p-t* at *k-t* but p is true (or false) therefore, it *must* (or must not) rain at *p-t*. The advocates of the three-valued logic may say that since *k-t* is not contemporaneous or posterior to *p-t*, p cannot be said to be true or false at *k-t*. Therefore p, by virtue

of its containing *p-t*, is only possible at *k-t*. As a consequence, in two-valued logic (1) may be interchanged with (2) or (3) (as a matter of fact, some persons have proposed this interchanging) without changing its truth-value. But in the three-valued logic, this interchanging is not permissible because in that case the truth-value of (1) is bound to change.

But, we may ask, when the value of (1) is possible, will it remain possible for ever, (just as if (2) or (3) is true, it will remain true for ever) or will it change subsequently? In the first case, we have an absurd situation where with regard to one and the same event one proposition is possible but the other proposition is true. Those who may maintain this position will have to assume that *k-t* of (1) is the same as *p-t* of (1). Had this not been so, then *k-t* being different from *p-t*, the proposition would remain true or false but my knowledge of it will be possible at *k-t*. But, then, it will not be the value of the proposition. If *p-t* of (1) is t_2 and its *k-t* is t_1, then there will be no difficulty in changing (1) into (2), because *p-t* of (1) and (2) may be the same. But in that case we shall have to say that although the truth-value of (1) at its *k-t* is *known* as possible yet from the point of view of *p-t*, (1) is intrinsically true. This will defeat the very purpose for which the three-valued logic was formulated.

If, on the other hand, the value of (1) is accepted to change subsequently, then at the time when (2) is true (1) will no longer be possible; it will also become true. This, again, is based on the assumption that *k-t* and *p-t* of (1) are the same. Only when we suppose that *k-t* and *p-t* of (1) is t_1 (i.e. (1) is known to be possible and in fact it is possible at t_1), then and then alone we can say that at t_2 (i.e. when it actually rains and (2) is true) (1) is no longer possible but true. In that case *p-t* of (1) and *p-t* of (2) are the same but *k-t* of (1) cannot be the same as *k-t* of (2). So when (1) is supposed to change its value subsequently, the change is being determined by change in *k-t*; *p-t* is completely ignored. But it cannot be appreciated that the proposition :

It will rain at t_2

changes into—

It rains at t_2,

merely because the fact of rain at t_2 is the same. Thus such a change will involve change in our knowing the value of a proposition but not in the value of the proposition itself. The proposition (in the case of (1) will remain true but my knowing makes all the difference between (1) and (2). If this alone is the result of the three-valued logic, then it is useless because the same purpose could be served by the two-valued logic also.

Let us also examine the nature of this third value. Is it the truth of a proposition which is regarded as possible at the time of making a statement or is it the event in question which is conceived as possible? If the logic of the word 'possible' gives any indication, in its adjectival form it is primarily used to mean that which may be or may happen or that which may be done. In this sense, we cannot say that the truth of a proposition may be or may happen or, for that matter, may be done. This word is also used to indicate something contingent and in that case a proposition having possible as its truth-value will ultimately mean that the truth of the proposition depends upon the happening of the event in question. Perhaps, in the three-valued system the word 'possible' is used in this sense. Here the possibility of any event happening need not engage our attention because philosophy or logic is not concerned with this possibility of events. Let us, therefore, examine whether the truth of a proposition can be conceived as possible when:

(i) [at t_1] $p=1/2$

(ii) [at t_2] $p=1$ or $p=0$.

Can we say that in spite of the variation in their respective values ps in both the cases are the same? If they are the same, then nothing can prevent

us from substituting (1) by (2) and the distinction befween true and possible propositions will be obliterated. If ps in the two cases are different and pv∼p is obsolete in the three-valued system, then we can have:

(a) p. ◇ p,

(b) ∼p. ◇ p (◇p=df. p is possible).

This means that at t_2 we can have both p and ◇p. But at t_2 p is true. Therefore, ◇p can be asserted only when (1) is not supposed to contain any reference to t_2. In that case, (1) will have to be expanded somewhat like 'At t_1 rain is possible.' It is so because if we say 'At t_1 rain is possible at t_2' then at t_2 (when rain actually takes place) ◇p no longer holds good. At that time rain is an actuality. Thus ◇p can have no reference to t_2 and it is not about t_2; it is about t_1. Maintaining p and ◇p as different, we cannot have the futurity of (1) in the sense in which we talk about the future. The only solution of this difficulty seems to be found in maintaining p as perpetually true or false and keeping *p-t* away from its confusion with *k-t*. We can at most say that a proposition conveyed by a future-tensed statement is true or false intrinsically but at the time the statement is made, its truth cannot be asserted because of the lack of knowledge of the event in question. But this obviously makes the system of three-valued logic superfluous.

III

In the case of propositions like (2) and (3), the event being either contemporaneous with or anterior to the assertion, the truth of an empirical proposition can be asserted on the basis of verification. But no such basis is available when a proposition conveyed by a future-tensed statement is sought to be asserted. It is impossible to assert a proposition without knowing that it is actually true. "With regard to the future." says Anscombe, "there is no performance that is to be allowed to qualify as knowledge."[1]

As knowledge involves certainty, no one can claim to know with certainty what is going to happen in the future. But there is a logical necessity in pv∼p in the two-valued system of logic. Therefore, even without being able to know what is going to happen, a person can say that an event is bound to take place and that nothing can prevent it. Thus, in this case there seems to be no escape from fatalism.

Professor Ryle[2] thinks that future-tensed statements cannot convey singular propositions; they can convey only general propositions. For him this is a logical necessity. Even though we may not be able to know what is going to happen, yet one thing is certain that propositions conveyed by future-tensed statements, of necessity, cannot contain any particular as their constituent. Proper names being demonstrative cannot be used in a situation where the object sought to be demonstrated is not yet in sight. Prof. Ryle would like to hold that the use of ordinary proper names is purely descriptive in future-tensed statements. Professor Ayer, on the other hand, thinks that "there is nothing in the least suspect about such singular propositions as 'this tree will bear fruit next summer'.........'Professor Ryle will be lecturing at 10.00 a.m. tomorrow morning'[3]." Suspecting that the subject in these propositions stands for demonstrable entities, he produces another example where the whole proposition is entirely about the future. He thinks that future-tensed statements can convey propositions which have particulars as their constituents although there may not be anything to be demonstrated.

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1. Aristotle and the Sea Battle, *Mind.* Vol. lxv, No. 257, Jan. 1956, Pp. 1-14.
2. What was To Be, *Dilemmas*, Pp. 26-27.
3. See chapter on Fatalism in the *Concept of a Person*, London 1963, Pp. 235-268.

I think that ordinary proper names are used in propositions which are entirely about the future descriptively. They are names without naming anything. It would be wrong to suppose, as Prof. Ayer supposes, that proper names in these propositions could be used significantly in the same sense in which they are used either in present or in past-tensed statements. Thus, it is not the presence or absence of knowledge that makes the use of proper names possible or impossible; it is rather the logical nature of proper names that prevents us from using them that way in propositions about the future.

We may agree with Prof. Ryle when he says that only those future-tensed statements can be significantly used that convey general propositions, although we cannot claim to know anything except that a proposition must be true or false. Does this conclusion support fatalism?

I think that if necessity which is found in pv∼p is regarded as in some way connected with the fact, then fatalism becomes inevitable. When (1) is said to be necessarily true, it is assumed that the fact that there will be rain and the truth of the proposition 'it will rain' have a necessary connection whereby the proposition is necessarily true. But this connection needs investigation. Is an empirical proposition necessarily true because there is a fact or is it the case that there is a necessary fact because there is an empirical proposition? I think that the very idea that a proposition is an empirical proposition compels us to accept the view that a proposition is necessarily true *because* there is a corresponding fact. In the cases of past and present events, the truth of a proposition and the fact in question are so intermingled that this question does not present itself to our mind although we are justified in saying that

> If and only if you are sitting on a chair then 'You are sitting on a chair' is true necessarily, but we cannot say: If and only if 'you are sitting on a

chair' is true necessarily then you are sitting on a chair.

It seems that there is a necessary relationship between the fact and truth of a proposition but I am inclined to hold that there appears no such necessary relationship between the truth of a proposition and the corresponding fact. When we conclude that fatalism is inevitable we assume, perhaps wrongly, that there is a necessity between the truth of a proposition and a fact.

Therefore, the kind of necessity which obtains in a true proposition will have to be distinguished from the kind of necessity which is found holding between a fact and a true proposition. The first one is the logical necessity which is to be found in p v~p but the second one seems to be a factual necessity. Therefore, in the case of future-tensed statements we cannot say, on the ground of the logical necessity of a proposition being either true or false, that existence of a fact corresponding to the proposition is inevitable. This conclusion may look odd but this oddity is felt because we are conditioned in our thinking by the age-old practice of entertaining the two necessities as of the same kind.

J.N. Kapur

AXIOMS OF INDUCTIVE LOGIC AND DECISION THEORY*

1. *Introduction*

ALL new knowledge is acquired by the alternate use of inductive and deductive logics. When man observes some facts of physics, chemistry, astronomy, economics, psychology etc., he uses inductive logic to set up some hypotheses to explain the observed facts. He deduces conclusions from the hypotheses postulated, by using the methods of deductive logic. He collects further facts, by experiments or observations, to check whether these conclusions are verified. If all the observations and results of experiments are verified, he accepts the hypotheses as basic laws and deduces further conclusions from the hypotheses and the theorems deduced from these hypotheses. If all these consequences are not verified, he uses inductive logic to modify the hypotheses and again proceeds to draw conclusions from these modified hypotheses. He continues to make alternate use of inductive and deductive logics till he arrives at some hypotheses which explain all the facts observed so far. These hypotheses are then considered to be the laws of the physical or social or biological sciences concerned.

Mathematics involves essentially deductive reasoning.

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It has been defined as 'If S then S_1 where S is a set of axioms and S_1 is a set of theorems. This essentially deductive tool, however, also plays an essential role in inductive logic. This is made possible by adding to the principles of deductive logic some 'axioms of induction' i.e. some principles which make inductive inferences possible. Once we accept these axioms, mathematics can be used to draw conclusions deductively.

There can be no 'proofs' for the axioms of induction except the 'a posteriori ones', viz. that we accept the axioms so long as their use goes on giving satisfactory results. In the present paper, we discuss some of these axioms. Mathematicians, statisticians, scientists and engineers have used these axioms to reap rich harvests and as such have not bothered so far to investigate deeply their foundations. The postition is similar to that of the set theory where the basic axioms gave very important and useful results which enriched mathematics in almost all directions. Later, however, some contradictions were discovered. These led mathematicians, philosophers and logicians to investigate the foundation of the axioms on which the set theory was based. It is desirable that mathematicians, statisticians, philosophers and logicians should undertake a large-scale investigation into the foundation of the axioms of induction which we are going to discuss below. They should not wait for serious contradictions to occur before agreeing to undertake this useful study.

The axioms of induction occur in the domains of mathematics itself, in statistics and in the decision theory. We shall discuss the first and second of these sets briefly and the third set, viz. the set of axioms of the decision theory, in greater detail.

2. *Induction Axioms in Mathematics*

An important axiom of fundamental importance in mathematics is the so-called *principle of finite induction*

which states :

"Let there be associated with each positive integer n, a proposition $f(n)$ which is either true or false. If firstly $f(1)$ is true and secondly for all k, $f(k)$ implies $f(k+1)$, then $f(n)$ is true for all positive integers n."

As an illustration, we consider the problem of finding the sum of first n positive integers. We find

$$(1)\quad 1 = 1 = \frac{1\times 2}{2}$$

$$(2)\quad 1+2 = 3 = \frac{2\times 3}{2}$$

$$(3)\quad 1+2+3 = 6 = \frac{3\times 4}{2}$$

$$(4)\quad 1+2+3+4 = 10 = \frac{4\times 5}{2}$$

These particular cases lead to the 'guess'

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

If we call this proposition $f(n)$, we know that $f(1)$ and also $f(2)$, $f(3)$ and $f(4)$ are true. In fact, we can verify the proposition for any positive integer n, but no amount of verification constitutes a proof. For this we appeal to the principle of induction. Assuming $f(k)$, i.e. $1+2+3+\ldots\ldots+k=\frac{k(k+1)}{2}$. We find whether it implies $f(k+1)$. We have $1+2+3+\ldots+k+(k+1)$ $=\frac{k(k+1)}{2}+k+1=\frac{(k+1)(k+2)}{2}$

Thus $f(k)$ implies $f(k+1)$ and therefore f (n) is true for all positive integers n.

Here we have assumed the truth of $f(k)$ in proving $f(k+1)$. We can assume the truth of $f(k)$ for all $k<n$. In fact, we have the *second principle of finite induction* which states :

"Let there be associated with each positive integer n a statement f (n). If, for each m, the assumption that $f(k)$ is true for all $k<m$ implies the conclusion that $f(m)$ is true, then $f(n)$ is true for all n."

An important generalisation of this is the principle of *transfinite induction* which states :

"Let X be a well ordered set and let $f(x)$ be a propositional function with arguments running through the set X and satisfying the following condition for every x;

If $\wedge y[y$ precedes $x \supset f(y)]$, then $f(x)$.

Then every element of the set X satisfies $f(x)$ i.e. $\wedge x f(x)$."

To understand the importance of this principle we make the following remarks :

(i) *y* precedes *x* is a propositional function of two variables.

(ii) This function is said to establish an ordering of set X if the following conditions are satisfied :

1. If *a* precedes *b*, then *b* precedes *a* does not hold.
2. If *a* precedes *b* and *b* precedes *c*, then *a* precedes *c*.
3. If $a \neq b$, then either *a* precedes *b* or *b* precedes *a*, e.g. $x<y$, $x>y$ gives ordering of the set of real numbers.

(iii) An ordering of a set X is said to be well-ordered if every non-empty subset of the set A has a first element, e.g., the set of all natural numbers is a well-ordered set.

(iv) In the case of the well-ordered set of natural numbers, the principle of transfinite induction reduces to the second principle of finite induction.

(v) The principle of transfinite induction is basic to the development of all modern mathematics and is typical of the principles which have to be added to a deductive system to make inductive inferences possible.

3. *Induction Axioms in Statistics*

In statistics, a given set of observations is regarded as a *random sample* from a population and our object is to draw as much as possible, ideally the whole of information about the population from a knowledge of the sample. The first step in this process is *specification*, i.e., we specify the population from which the population could have come. For this purpose, there is no guidance except intuition and intimate knowledge of the field to which the sample belongs. The statement of the population involves some parameters which have to be estimated. This is the problem of *estimation*. For this purpose we have the following alternative axioms of induction.

(i) *the principle of maximum likelihood* which states that the best estimators are those which maximize the likelihood of the sample having come from the population.

(ii) *the principle of least squares* which states that the best estimators are those which minimize the sum of squares of the differences between observed values and expected values.

(iii) *the principle of inverse probability* which states that the best estimators are those which maximize the 'a-posteriori' probability of the sample having come from the specified population.

(iv) *the principle of minimum chi-square* which states that the best estimators are those which minimize the sum of squares of the differences between observed and expected frequencies divided by expected frequencies.

(v) *the principle of moments* which states that the best estimators are obtained by equating the necessary number of moments of the sample with the corresponding number of moments of the population.

(vi) *the principle of unbiased minimum variance* *estimators* which states that the best estimators must be such that their expected value is equal to the population parameter which is to be estimated and its variation in different samples is as small as possible.

To find out the best principle, we consider Fisher's criteria for best estimators, viz. *consistency*, *efficiency* and *sufficiency*. The method of maximum likelihood is found to be the most satisfactory from this point of view. However, some of the above axioms and their applications have been subjects of great controversies in statistics.

Sometimes instead of estimating the parameter, we *test* some *hypothesis* about the parameter. Since we have knowledge of only a sample, we can make two types of errors in testing a hypothesis, viz:

(i) we may reject the hypothesis when it is true.

(ii) we may accept the hypothesis when it is false.

Thus a judge, on the basis of the evidence, may convict an innocent person or acquit a guilty person or an examiner may pass an undeserving candidate or may fail a deserving candidate.

The two types of errors are called *first and second kind of errors*. We shall like to reduce both types of errors as much as possible, but for a sample of a fixed size this is not possible. The axiom which Pearson and Neyman adopted was the following:

> "Keeping one type of error fixed, that testing procedure is best which minimizes the other type of error."

We can keep both types of errors at less than a fixed level, by adjusting the sample size. We draw one observation and decide either to accept the hypothesis or reject the hypothesis or continue observations. We continue this procedure until we either accept the hypothesis or reject the hypothesis. This idea is basic to *sequential analysis* first investigated by Abraham Wald.

4. *Structure of Decision Theory*

At its earliest stages, a science is purely descriptive, then it becomes predictive and finally it tries to control physical or biological or social phenomena for the benefit of mankind. All problems of science are thus problems of description, prediction and control. Further, all problems of control are decision problems. It is desirable that we should be able to control the outcome by suitable decisions.

We can make good decisions, but good decisions may or may not lead to good outcomes, just as bad decisions may or may not lead to bad outcomes. The decision problems may arise in military or industrial fields or anywhere else.

Uncertainty is an essential element of the decision theory. Pure certainty is trivial. If all factors are known with certainty, decision making is a comparatively simple and uninteresting problem. Further the measure of uncertainty we use has to be a real number so that it may be compared with other uncertainties.

The components of a satisfactory decision theory are:

(i) *a satisfactory theory of probability*

We must consider probability as a state of mind and not as an abstract mathematical entity with an independent existence. An example of this is provided by the consideration as to how the probability of successful working of a given machine is changed by the failure of other similar machines. The machine itself does not change, so the probability should not change, but probability being a state of mind is influenced by what happens with other similar machines.

(ii) *A satisfactory way of specifying the objective function* which is meant to be optimized by the decision process.

(iii) *A successful theory of constructing criteria for* *comparing decisions in terms of measures of risk.* The risk a man is prepared to run depends on the circumstances. A man may be prepared to play an even bet if it concerns ten rupees, he may not agree to it if it concerns his entire fortune. A man may be prepared to take a risk if there is a five per cent chance of his losing some money; he may not agree to take a risk if there is a five per cent chance of his losing his life.

5. *An Example of a Quantitative Decision Process*

A judge can convict or acquit a person. On the basis of the evidence supplied, he estimates the probability of guilt to be p. He can take two decisions. To compare the two decisions, he compares the expected cost to the society of the two decisions and decides in favour of the decision which gives smaller expected value. Let

A = prisoner's contribution to society if released.

B = cost of imprisonment.

C = indemnity to the prisoner if he is found innocent after being imprisoned.

D = cost of crimes to society if a guilty man is released.

There are four alternatives and their costs to society are as follows:

(i) the cost of imprisoning a guilty person is $+B$.

(ii) the cost of imprisoning an innocent person is $+B+C$.

(iii) the cost of freeing a guilty person is $-A+D$.

(iv) the cost of freeing an innocent person is $-A$.

Expected cost to society of declaring the person guilty $= +pB+(1-p)(B+C) = B+(1-p)C$.

Expected cost to society of declaring the person innocent $= -p(A-D)-(1-p)A = -A+pD$

Therefore the person should be convicted if

$$B+(1-p)C < -A+pD, \text{ i.e., if } p > \frac{A+B+C}{C+D}.$$

In the same way if we have a number of alternative decision possibilities and we can work out the expected costs of all the decisions, we can find the decision with the minimum expected cost.

6. *Statistical Decision Theory*

Suppose we are given a sample from a population and we are interested in the proportion of defectives in the population. (1) We may try to estimate the proportion p of defectives in the population. This is the problem of estimation. (2) We may try to test the hypothesis that p is less than or equal to some number p_0. This is the problem of *testing of hypotheses*. (3) We may have to take some decisions, e.g., we may decide to reject the lot if p is greater than or equal to p_0, sell it at reduced price if p lies between p_0 and p_1 and sell it at full price if p is greater than or equal to p_1. This is the problem of *statistical decision theory.*

We have already studied the axioms necessary for inductive inferences connected with estimation and of testing of hypotheses. We now consider the axiom introduced to deal with problems of decisions.

We now introduce the following definitions:

(i) the set of all possible samples is called the *outcome space* Z.

(ii) the set of all possible values of the parameters is called the *parameter space* Ω.

(iii) the set of all possible probability distributions with parameters in the space Ω is called the space P of *probability* distribution.

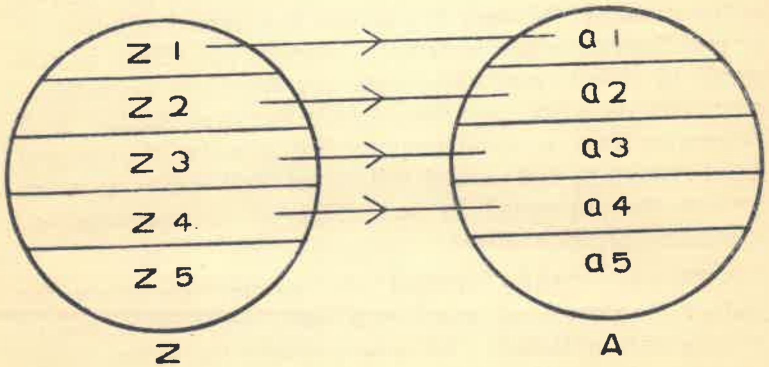
(iv) the set of all possible decisions or actions is called the *action space* A.

(v) in the light of the sample value z, we recommend a certain action $a = d(z)$ depending upon z and belonging to A. This function maps Z into A and is called a *decision function.*

(vi) for each w in Ω and for each a in A there is a *loss function* L (w, a) which expresses quantitatively the appropriateness of action a for the given w.

(vii) the expected value of L (w, d (z)) is called the *risk function* and is denoted by R (w, d).

Our object is to partition the outcome space Z into disjoint subsets and to assign a different particular action for each such subset.



If the outcome lies in Z_1, we recommend action a_1. This partitioning or mapping is called a *statistical decision procedure*. To evolve this procedure, we regard it as a game between us and Nature. Nature is allowed to specify w on which the partitioning depends. What is the principle according to which Nature chooses w?

It is here that we introduce the axiom which we may call the *principle of perversity of Nature*. We assume that Nature would always do its worst and for any decision rule d we choose, it would choose w so as to maximize the risk function R (w, d). Let this value be called w_d. Now consider the set of functions R (w_d, d). These depend on the decision rule we evolve. We now choose d so as to minimize R (w_d, d). The resulting decision rule is called the *minimax decision rule*.

As usual, there can be no proof for an axiom like this. However, some persons threw pieces of bread buttered on one side a large number of times and found that the buttered face came downwards in a majority of cases and from this concluded that the perversity of Nature was verified!

The minimax decision rule minimizes the maximum possible risk. The rule is a rule of overcaution, but mathematically it is easier to build up the theory for the case of complete opposition by Nature than for the case of partial opposition.

7. *Theory of Games*

We have considered above decision processes as games against Nature. Similar decision processes arise when we consider games between two persons, say, A and B. A has a choice of M moves and the probability that he makes the ith move is p_i ($i=1, 2, — — —$ M). Similarly B has a choice of N moves and the probability that he makes the jth move is $p_j (j=1, 2, —, N)$. If A makes the ith move and B makes the jth move, A receives the quantity a_{ij} and B receives the quantity b_{ij}. The matrices (a_{ij}) and (b_{ij}) are called the *pay-off matrices*. A special case of interest arises when $b_{ij} = -a_{ij}$, so that B pays what A receives. The situation we are considering is called a *two-person zero-sum game*. We can also consider many-person and non-zero sum games, but their theory is not so well developed as that of two person zero-sum games.

The expected return for A is

$$E_A \; (p, q) = \sum_{j=1}^{N} \sum_{i=1}^{M} a_{ij} \, p_i \, q_j,$$

and the expected return for B is the negative of this.

The axiom that we introduce is that A wants to maximize his own expected return and that B wants to maximize his own. In view of our zero sum hypothesis, this implies that A chooses his strategy so as to maximize $E_A(p, q)$ and B wants to minimize it.

If A is asked to choose his strategy before B, he argues as follows:

Whatever strategy (i. e., probability distribution) A chooses, B will choose his strategy so as to give the minimum return to A. There will be different minima for A for the different strategies he chooses. He now chooses his strategy so as to maximize his own return. Thus his expected return would be

$$\underset{(p)}{\text{Max}} \quad \underset{(q)}{\text{Min}} \quad E_A(p, q)$$

Similarly if B is asked to choose his strategy before A, the expected return for A would be

$$\underset{(q)}{\text{Min}} \quad \underset{(p)}{\text{Max}} \quad E_A(p, q)$$

In each case the variation is taken over the regions determined by the relations

$$p_i \geq 0, \sum_{i=1}^{M} p_i = 1; \quad q_j \geq 0, \sum_{i=1}^{N} q_j = 1$$

It appears that the return for A depends on whether he is allowed to make the move first or not, but this is simply not true, for Von-Neumann proved that

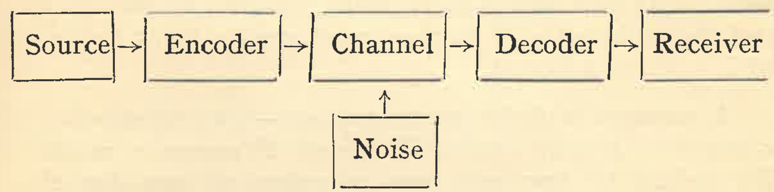
$$\underset{(p)}{\text{Min}} \quad \underset{(q)}{\text{Max}} \quad E_A(p, q) = \underset{(q)}{\text{Max}} \quad \underset{(p)}{\text{Min}} \quad E_A(p,q)$$

This min-max theorem of Von-Neumann is basic to the theory of games. This implies that A can announce his probability distribution without giving the slightest advantage to B and the same holds true for B. This astonishing result, achieved by introducing the concept of strategies, i.e., of choices of probability distribution for A and B, removes the difficulty involved in possible non-simultaneous moves by A and B.

The common value of the two expressions above is called the *value of the game* and is usually denoted by v or by $v(a_{ij})$.

8. *Information Decision Processes*

A communication process can be represented as follows:



An example is the telegraph system. The source is the ordinary English (or Hindi) language. The encoder transforms the English alphabet into the language of dot, dash and space. The channel transmits the encoded message which is decoded by the decoder and sent to the receiver.

Important characteristics of almost all communication systems are:

(i) the source is stochastic, i.e., the messages coming from the source can be described by a probability distribution.

(ii) there is noise in the channel, i.e., the messages coming out of the channel are distorted versions of the messages sent into the channel.

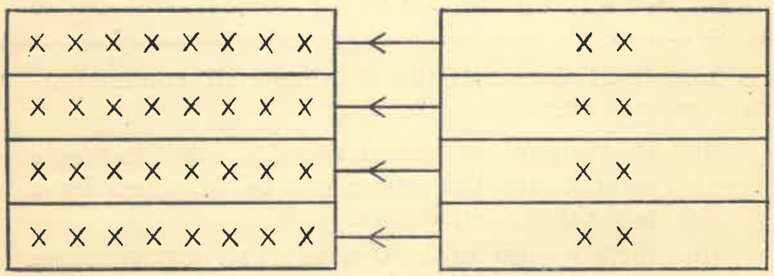
(iii) The noise characteristics of the channel are described in terms of probability distributions.

Our problem is the following:

Given the above probability distribution and given the distorted messages, to detect errors and correct them, i.e., to decide the messages transmitted. Of course, in each case we cannot find the correct message sent, but we can find the message with the greatest possibility of having been sent. Thus, we have to consider decision processes of a stochastic nature.

The most important channel is the so-called binary symmetric channel. Here only two symbols 0 and 1 are transmitted. A zero or one may be received as a zero or as a one. The probability of correct symbols being received, i.e., the probability of a zero being received as zero and of a one being received as one is p and the probability of an inncorrect symbol being received is q where $p+q=1$.

A message is of the form $(a_1, a_2, —, a_n)$ where each a_i can be a zero or a one. There are 2^n messages in all. We decide to transmit only a subset of, say, d or 2^k messages where $k< n$. This subset is called a *code*. The outcome space consists of all the 2^n messages and we have to partition this outcome space into 2^k disjoint subsets such that whenever a message is



2^n messages received 2^k messages sent

received, we have a decision rule by which we decide as to which of the 2^k messages was sent. The problem is similar to the one we had faced when we discussed the statistical decision theory. The parameter p is fixed and we choose our rule in such a manner that the probability of correct decoding is maximized. This is the axiom of the present system. A good deal of work has been done to develop codes satisfying this axiom, but the

complete answer is not yet known. An alternative axiom for this decision process is that the decision rule should be such as to ensure correction for a number of errors upto a fixed number, say t. A more recently developed axiom which combines these two is that the decision rule should be such that it should correct upto t errors and maximize the probability of correct decoding. The mathematical consequences of these axioms are still being developed.

9. *Bayesian Inference*

In the communication system discussed above, the parameter p viz. the probability of correct transmission was supposed to be known. In practice this is not so. At most we know some probability distribution for p. As more and more messages are transmitted across the channel, it may be possible to use the knowledge of distortions of these messages to improve the knowledge of probability distribution of p. The theorem which enables us to do this is called Bayes theorem. Given the 'a priori' probability distribution of p, i.e., the probability distribution of p before the additional information is available, this theorem enables us to find the 'a posteriori' probability distribution, i.e., the probability distribution of p after this additional information is available.

A similar problem arises in inventory control of items of a new style. In the beginning of the season, the shopkeeper has some *a priori* probability distribution for the proportion p of customers who will be buying this new style of goods. As the season progresses and he gets more and more information on sales of goods, he can use Bayes theorem to improve the probability distribution for p and this can influence his placing of new orders for this type of goods.

Sometimes the '*a priori*' information is altogether absent. In the absence of this information we assume a rectangular distribution, i.e., we assume that all values of p are equally likely. This will be the Bayes axiom of the system and the results of using this axiom will be as true or as false as the axiom itself. In the past quite a number of controversial decision processes have been solved on the basis of this axiom.

The principle of feeding back additional information obtained in the course of an operation to improve the working of the future operation is a significant development of modern times. The feedback control systems or adaptive control systems are based on this principle. Bayes theorem, if correctly applied, is a powerful tool for these systems.

10. Dynamic Inference

An important class of decision problems arising in mathematics, science, economics, industry and technology are the so-called *multistage decision problems* where some *decision variables* are to be chosen in each of the n stages. The sequence of decisions is called a *policy* and our object is to choose an *optimal policy*, i.e., a policy which optimizes some *objective function* subject to some given *constraints*. Thus, we may want to choose decisions every year so as to maximize the national income over the next twenty years subject to the constraints imposed by limited resources in materials and trained manpower.

The basic axiom for this purpose is Bellman's *principle of optimality* which states that for an optimal policy for an n-stage process, whatever be the initial decision, the remaining n—1 decisions must be chosen so as to constitute an optimal policy with respect to the state resulting from the first decision.

This principle enables us to express the outcome of an n-stage optimal policy in terms of the outcome of an (n—1)—stage optimal policy. The outcome of a one-stage optimal policy can usually be easily found. By the principle of finite induction we can then find the outcome of the n-stage optimal policy for all n.

An n-stage optimization problem usually involves maximization or minimization of a function of n variables. The principle of optimality reduces this to n optimization problems for functions of one variable. This reduces the complexity of the problem from the order of about n to that of about n. For large values of n this involves a tremendous simplification, as can be easily verified.

11. Summary of Axioms of Inductive Logic

The axioms of induction discussed in the paper are summarised below :

(i) principle of finite induction (§ 2)

(ii) second principle of finite induction (§ 2)

(iii) principle of transfinite induction (§ 2)

(iv) principle of maximum likelihood (§ 3)

(v) principle of method of moments (§ 3)

(vi) principle of unbiased minimum variance estimators (§ 3)

(vii) principle of minimum chi squares (§ 3)

(viii) principle of inverse probability (§ 3)

(ix) principle of least squares (§ 3)

(x) the principle that one type of error being fixed, that test is best which minimizes the other type of error (§ 3)

(xi) the principle that both types of error being fixed that test is best which minimizes the expected sample size (§ 3)

(xii) the principle that for a number of alternative decisions, we should choose that one which gives the minimum expected cost (§ 4)

(xiii) the principle that for a number of alternative decision rules, that one which minimizes the maximum risk should be chosen, the maximum risk occurring when Nature does her worst (§ 5)

(xiv) the principle that for a zero-sum two-person game, the optimal strategy for each player arises when each tries to maximize his gain (§ 6)

(xv) the principle that that coding is best which maximizes the probability of correct decoding (§ 7)

(xvi) the principle that that coding is best which corrects up to *t*-errors in transmission (§ 8)

(xvii) the principle of optimality for multi-stage decision processes (§ 10)

12. Concluding Remarks

We once again express the hope that mathematicians, statisticians, engineers. logicians and philosophers will cooperate together to examine the foundations of the axioms of induction summarised above.

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ON DIFFERENT APPROACHES TO THE THEORY OF PROBABILITY

THE article presents a discussion on subjective and objective probabilities largely due to Keynes. It indicates how subjective element enters in the latest definition of objective probability, due to Kolmogorov, likewise in earlier definitions of objective probability. The famous controversy, arising out of the above subjective element, between Fermat and Roberval, the French mathematicians of 17th century, is presented and an experimental solution of the controversy is provided.

1. Introduction

The word 'proposition' as used in ordinary conversation may be applied to any word or phrase which conveys information of whatever kind. Example—The words "Yes or No" are propositions in the ordinary sense of the word and so are the phrases—"You owe me Rs. 5," "I do not."

We may be said to know a proposition when we have a rational belief in it. We may have rational belief in the proposition either with certainty or without certainty. Thus our knowledge is either certain or a term meaning without certainty. When we have reasons not to believe in a proposition, the proposition is said to be improbable. If we have rational belief of any other degree, without certainty, in the proposition, the proposition is said to be probable. Symbolically, we can present the above discussion as follows.

Let there be a proposition *p*. Let an auxiliary proposition *q* state, on the basis of evidence *h*, that we have a degree of belief '*a*' in *p*. Then

$$q : P(p/h) = a.$$

When the proposition is certain, $a=1$ and when the proposition is improbable, $a=0$. For the rational belief, of any other degree, in the proposition p, $0<a<1$. The degree of rational belief in the proposition p, measured by the quantity a, on the basis of evidence h, may be said to be the probability of the proposition p given the evidence h.

Direct and indirect knowledge about p under suitable conditions leads to rational belief in p of an appropriate degree. It is not always possible, however, to analyse the mental process in the case of indirect knowledge or to say by the perception of what logical relation we have passed from the knowledge of one proposition to knowledge about another. The perceptions of some relations of probability may be outside the powers of some or all of us. What we know and what probability we can attribute to our rational beliefs is, therefore, subjective in the sense of being relative to the individual. But given the body of premises which our subjective powers and circumstances supply to us and given kinds of logical relations upon which arguments can be based and which we have the capacity to perceive, the conclusions (here regarding the degree of belief in a proposition), which itself, a rational for us to draw, stand to these premises in an objective and wholly logical relation. Our logic is concerned with drawing conclusions by a series of steps of certain specified kinds from a limited body of premises.

2. *Subjective and Objective Probabilities*

The early writers on probability, perhaps because of above considerations, have suggested that there are two kinds of probabilities: (i) Subjective probability and (ii) Objective probability. The following Art. 4, p. 284, in part, of Keynes is reproduced below:

"In the writings of Condorcet, I have said above, all is confused. But in Bertrand's criticism of him a relevant distinction, though not elucidated, is brought before the mind. "The motive for believing", wrote Condorcet, "that, from ten million white balls mixed with one black, it wlll not be the black ball which I shall draw at the first attempt is of the same kind as the motive for believing that the sun will not fail to rise to-morrow." "The assimilation of the two cases," Bertrand writes in criticism of tho above, "is not legitimate: one of the probabilities is objective, the other subjective. The probability of drawing the black ball at the first attempt is 1/10,000,000, neither more nor less. Whoever evaluates it otherwise makes a mistake. The probability that the sun will rise varies from one mind to another. A scientist might hold on the basis of a false theory, without being utterly irrational, that the sun will soon be extinguished; he would be within his rights, just as Condorcet is within his; both would exceed their rights in accusing of error those who think differently."

I further reproduce Keynes's Art. 6, p. 286 in part:

"Now a careful examination of all the cases in which various writers claim to detect the presence of 'objective chance' confirms the view that 'subjective chance,' which is concerned with knowledge and ignorance, is fundamental, and that so-called 'objective chance,' however important it may turn out to be from the practical or scientific point of view, is really a special kind of 'subjective chance' and a derivative type of the latter. For none of the adherents of 'objective chance' wish to question the determinist character of natural order; and the possibility of this objective chance of theirs seems always to depend on the possibility that a particular kind of knowledge either is ours or is within our powers and capacity."

Thus Keynes maintains that even the objective probability is a special case of subjective probability. In my opinion, the subjective character of probability is found in the subsequent approaches to the theory of probability. I illustrate this point by considering in detail the latest approach, due to Kolmogorov, to the theory of probability. I further illustrate how this approach, though subjective in character, helps us to arrive at certain inferences and then making certain decisions which are more rational in character than otherwise possible.

The Kolmogorov approach to the theory of probability gives rise to the following definition of probability for the finite and discrete cases. The theory of statistical inference or decision is based on this definition.

3. *Definition of Probability*

Let a given experiment E, performed under specified conditions, result in n outcomes, where n can be any positive integer from 1 to ∞. Let these possible outcomes be referred to as elementary events (or cases). It may be noted that these elementary events are mutually exclusive. Let us associate a non-negative number p_j $(0 \leq p_j \leq 1)$ to elementary event e_j such that

$\sum_{j=1}^{n} p_j = 1$ Let a given event A consist of m of these elementary events $e_{(1)}, e_{(2)}, \ldots, e_{(m)}$. Then by definition the probability of A is given by

$$P(A) = p_{(1)} + p_{(2)} + \cdots p_{(m)} \qquad (3.1)$$

The non-negative numbers p_j's are sometimes referred to as the probability of the elementary events e_j. It may be noted that $0 \leq P(A) \leq 1$. Since the experiment E consists of all elementary events, we may take $P(E) = 1$.

When both n and m are finite and when experience

suggests that p_j's are equal, i. e. the elementary events are equally likely; the probability of the events is simply given by

$$P(A) = \frac{m}{n} \qquad (3.2)$$

It may be noted that the subjective element in the definition appears in choosing p_j's. How this subjective element caused a controversy between Fermat and Roberval, the French mathematicians of 17th century, is discussed in Example 3.2 below. The two experiments are then presented to indicate how to resolve the controversy.

It may be further noted that in our definition there is no scope of finding probability of statements such as 'Sun rising tomorrow'; because such statements do not tell us how the experiment is to be performed which would result in 'Sun rising tomorrow', let alone the conditions of the experiment. We work out the following Example 3.1 to illustrate the above definition:—

Example 3.1—An unbiased coin is tossed once. Find the probabilities for the occurrence of a head and of a tail. Solution:—

Experiment: It consists in tossing the unbiased coin once.

The possible outcomes of the experiment are

(1) The occurrence of the Head which we denote by 'H',

(2) The occurrence of the 'Tail' which we denote by 'T'. Then we have elementary events: H and T. Since the coin is unbiased, the elementary events are equally likely. Let,

A: The occurrence of a Head.

B: The occurrence of a Tail.

Then,

$$P(A)=\tfrac{1}{2},\ P(B)=\tfrac{1}{2}.$$

We present the Example 3.2 and its two different solutions, first due to Fermat and the second due to Roberval.

Example 3.2—A throws a coin and then B throws. A wins if head comes. Calculate the probability of A's winning.

Solution : (i) Fermat's solution—The possible elementary events are HH, HT, TH, TT. These events are equally likely. The number of events in which at least one head appears is 3. Therefore the required probability is 3/4.

(ii) Roberval's solution—Since A wins when A gets a head. The elementary events can be taken as H, TH, TT. If these events are equally likely then the required probability is 2/3.

The difficulty in choosing between the two solutions was referred to by D. Alembert in his more than one publications (Cf. Todhunter). On the whole, he seemed to agree with Roberval's solution. We discuss below the two experiments carried out by nine students of the Department of Statistics, University of Rajasthan, in 1965-66. Each of the nine students performed first the following experiment :

Experiment 3.1. With a given four-anna coin obtain (a) Five series of 10 throws, (b) Five series of 100 throws and (c) Five series of 1000 throws. For each of the fifteen series, find the proportion of heads obtained in the series. Write down the maximum and minimum of proportions for the series in (a), (b) and (c) and then comment whether your coin is unbiased or not.

Then each of the nine students performed the following second experiment by choosing another partner of their own :

Experiment 3.2. If the experimental evidence indicates that your four-anna coin is unbiased, then you

throw the coin first and then ask your partner to throw it. The game is won by you if at least one head appears. A person X says that the chance (as commonly understood) of your winning is 3/4. Another person Y says that the chance of your winning is 2/3. Proceeding on the lines of Experiment 2.1, decide whether you should agree with X or Y or with neither.

The results of the two experiments are given in Tables I and II of Appendix I. From Table I, it appears that each of the nine coins used by the students is unbiased. Further from Table II, it appears that one should choose Fermat's solution over Roberval's solution in Example 3.2. Whether the degree of approximation in saying that the coins are unbiased is well within the permissible limits of sampling error can be found by noting that the distribution of proportion of heads in n throws of the coin for $n \geq 20$ is normal as seen from the graphs plotted in Appendix II. The normal theory, then, provides the solution to the problem. A similar method will provide the necessary device for measuring the appropriateness of the approximation involved in deriving the conclusion from Table II.

My thanks go to Shri N.K. Bapna and Shri Hari Narain Sharma of the Department of Statistics for assisting me in the preparation of the paper.

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Appendix I

Results of the experiments 3.1 and 3.2 performed by nine students of MA/MSc (Previous) of Department of Statistics, University of Rajasthan (1965-66)

Table I

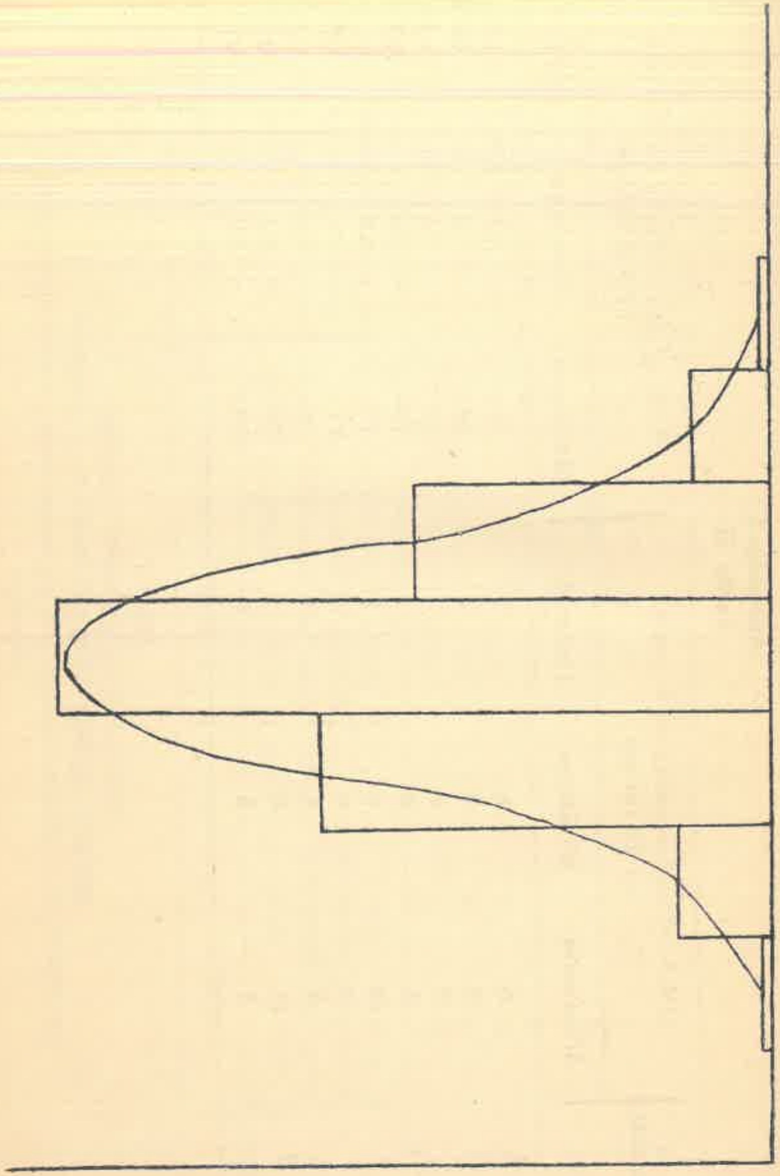
Students	5 Series of 10 throws			5 Series of 100 throws			5 Series of 1000 throws		
	Maximum proportion of heads	*Minimum proportion of heads*	*Difference*	*Maximum proportion of heads*	*Minimum proportion of heads*	*Difference*	*Maximum proportion of heads*	*Minimum proportion of heads*	*Difference*
A	.6	.5	.1	.53	.47	.06	.505	.485	.020
B	.6	.4	.2	.55	.46	.09	.515	.470	.045
C	.6	.4	.2	.50	.47	.03	.501	.490	.011
D	.6	.4	.2	.55	.46	.09	.511	.478	.033
E	.6	.3	.3	.54	.47	.07	—	—	—
F	.5	.3	.2	.51	.47	.04	.510	.490	.020
G	.5	.2	.3	.56	.45	.11	.552	.500	.052
H	.7	.6	.1	.58	.47	.11	—	—	—
I	.7	.4	.3	.54	.48	.06	—	—	—

Appendix I

Table II

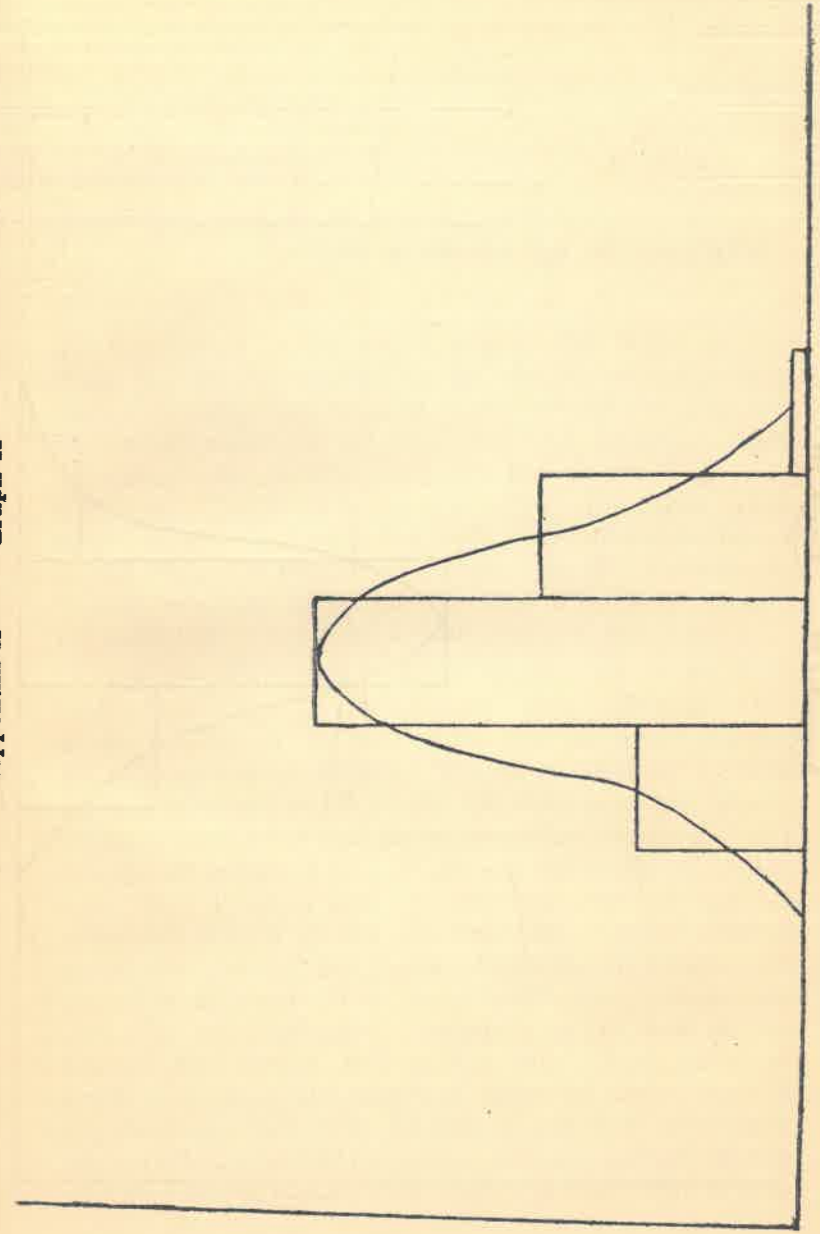
Students	Prob. of self-winning in 5 series of 10 throws			Prob. of self-winning in 5 series of 100 throws		
	Maximum	*Minimum*	*Difference*	*Maximum*	*Minimum*	*Difference*
A	.8	.6	.2	.76	.74	.02
B	.8	.6	.2	.81	.75	.06
C	.9	.6	.3	.76	.74	.02
D	.8	.6	.2	.76	.73	.03
E	.9	.8	.1	.79	.72	.07
F	.8	.6	.2	.81	.74	.07
G	.8	.5	.3	.77	.74	.03
H	.8	.6	.2	.76	.74	.02
I	.9	.8	.1	.79	.72	.07

Appendix II Graph I



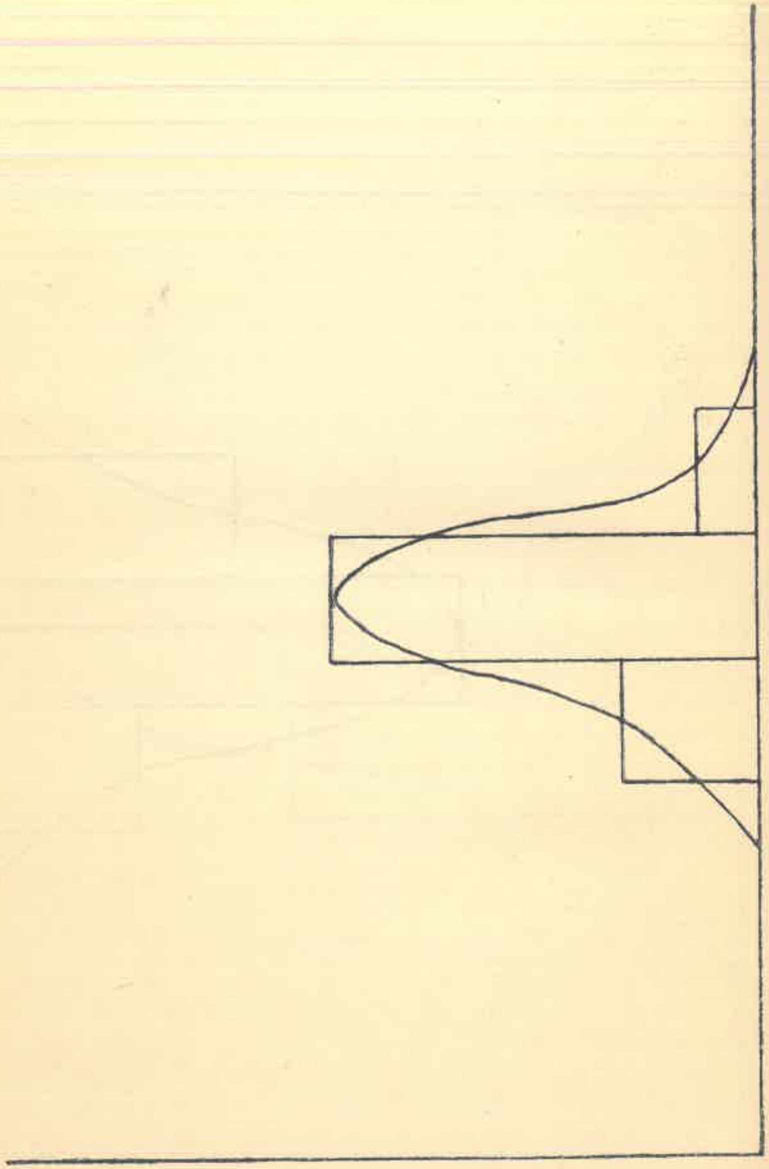
Graph for proporti on of heads in 500 series of 10 throws of a four-anna coin (Data due to H.N. Sharma)

Appendix II Graph II



Graph for proportion of heads of 250 series of 20 throws of a four-anna coin (Data due to H.N. Sharma)

Appendix II Graph III



Graph for proportion of heads of 165 series of 30 throws of a four-anna coin (Data due to H.N. Sharma)

R. Shukla

THE NATURE OF INDUCTION

IN the history of philosophy, one of the main sources of pseudo-philosophies is the asking of wrong questions. Buddha used to keep quiet whenever questions were asked which did not arise from the context. Such questions were looked upon by him as meaningless or irrelevant. It seems to me that asking the right questions should be one of the main philosophical activities in any branch of human knowledge. In this short paper, I shall endeavour to pursue this kind of activity regarding induction (*Vyapti,* as Indian logicians would put it).

For this discussion, it will be worthwhile to go back to the Indian system of reasoning. Let us imagine that we are seeking an explanation for the mountain having fire (*Pratijna* or assertion). We try to advance grounds for this inference which are (i) the mountain has smoke (*Hetu* or condition) and (ii) whatever has smoke has fire (*Vyapti* or general law). It goes to the credit of Indian logic that it took care to see that the explanation advanced is not *ad hoc* but scientific. And it insisted upon the general law being verifiable in at least one situation different from the present one. Hence the necessity of *Udaharana* (example) in the form of 'this kitchen has smoke and it has fire.' And, lastly, we apply (*Upanaya*), the admitted rules of deduction, and get the proposition 'the mountain has fire' (*Nigaman,* warranted assertibility). Here let us parenthetically add that this account of the theory of knowledge is complete and satisfying even from the point of view of the present day status of scientific reasoning.

Then the question that was asked is this: What are the rules of induction (i.e. how do we come by general laws of nature (*Vyapti*))? The *Nyaya* system, or for that matter the Western logicians, Mill and Bacon, conceded the reasonableness of this question (to which we shall come back a little later). And it is a part of philosophical history how heroic attempts were made to set up the canons of induction—the dominant idea being that the laws are generalisations from particular observations. It was Hume who conclusively established that no amount of particular observations can entail a general law. I shall make only two remarks to strengthen Hume's views. Suppose someone allows us to observe the formation of a few terms 2, 4, 6, 8 (in this order) and asks us to infer from these particular observations a general law of formation of a sequence whose first four terms are these. Now a simple-minded law that suggests itself to us is that the general law is:

t_n (the *n*th term) $=2n$,

according to which the fifth term would be 10.

But $2n+(n-1)(n-2)(n-3)(n-4)$ may be also taken as an expression for the general term t_n leading to the fifth term being 34. This shows that any number of laws can be framed to take care of the particular observations.

The second remark that I wish to make is that some of the laws suggested by particular observations are influenced by the language we use; while, if we operate in a richer language, a different law may be suggested. For example, observing that cows we have so far met are herbivorous, we may make a general law. 'All cows are herbivorous.' Now, suppose in our dictionary, there were a concept (i.e. term) like 'cherbivorous' where cherbivorous by dictionary meaning is 'herbivorous in the twentieth century and non-herbivorous in the twenty-first.' Then, from the same observations about the cows

as before, one can equally legitimately generalise that "all cows are cherbivorous.' But the two generalisations surely conflict.

Now, if we turn to the question, "What are the rules of induction?" we find that there are two assumptions which sustain it. The first assumption is that there are laws in Nature (in other words, there is order in Nature) and the second assumption is that there are logical rules of discovering them. It goes to the credit of Kant, roused from dogmatic slumber by Hume, that he recognised the unreasonableness of the assertion that there is order in Nature. Kant maintained that there is no order (or law) in Nature. Laws are our own mental constructs imposed upon Nature. And the mind arrives at these laws by means of its own built-in-mechanism in terms of forms of intuition. In other words, the laws are the questions that we put to Nature and they are not things extracted from Nature by laborious observations and experiments. But then the pertinent question is that if the laws of Nature are the laws of our mind, why did the mind not operate before Newton, for example?

Kant was right when he said that there is no order in Nature, but he was not quite right when he said that order exists in the mind. Thanks to Einstein and the ensuing freedom of thought from the stupefying influence of the Newtonian tradition, we now realise that there is order neither in Nature nor in the mind. Hume's question and Kant's answer are both mistaken. All that we do in building up theories (or laws) is to make bold conjectures and try them out in Nature. Most often we are unsuccessful in the sense that these conjectures are refuted by experience. But it is all luck when we stumble upon a conjecture which stands the test of refutation. Hence the right question to ask would be, "how successful conjectures are possible?" The answer is that we freely invent stories, myths or theories and try them out in Nature. The next pertinent question would be, "how theories are confirmed?"

Before discussing confirmation, let me in passing remark that this concept is very much related to the concept of theoretical testability of a sentence by means of empirical evidence and this in its turn illuminates what positivists call 'meaningfulness.'

Now regarding confirmation Nicod advances, let us say, 'entailment theory.' According to Nicod, a theory or a general law T: A implies B is said to be confirmed by an evidence *e* if *e* consists in observing B in a case of A. We wish to examine this simple-minded view of confirmation in terms of logical symbolism. We first express T as: $(x)\ (P(x) \rightarrow Q(x))$ i.e. "for any object, *x*, *x* is a P implies *x* is a Q." An object *a* will confirm T if it satisfies both P and Q and it will disconfirm T if it satisfies P but not Q; and it is neutral if it does not satisfy P. Now consider the general theories T and T'

T: $(x)\ ((\text{raven}\ (x)) \rightarrow (\text{black}\ (x)))$
All ravens are black

T': $(x)\ (\sim(\text{black}\ (x)) \leftarrow\rightarrow \sim(\text{raven}\ (x)))$
Whatever is not black is not raven

Now let us consider four objects *a*, *b*, *c*, *d*, such that

1. *a* is a raven and black;
2. *b* is a raven but not black;
3. *c* is not a raven but black; and
4. *d* is neither raven nor black

In view of Nicod's criticism, *a* confirms T but is neutral with respect to T'; *b* disconfirms both T and T'; *c* is neutral to T as well as T'; and *d* confirms T' but is neutral to T.

But T and T' are logically equivalent—they have the same content but different formulations. This fact that *a* and *d* confirm one and are neutral to the other is repugnant to our thinking. Hence we discussed Nicod's criterion. The weakness of Nicod's criterion can be best realised by putting T in the equivalent form T''. : (x)

$((((\text{raven}\ (x))\ \&\ \sim(\text{black}\ (x))) \leftarrow\rightarrow ((\text{raven}\ (x))\ \&\ \sim(\text{raven}\ (x))))$. But no object can satisfy both the antecedent and the consequent, simply because the consequent is a contradiction.

Because of these considerations we reject Nicod's criterion and demand that every confirming evidence of a theory must confirm it in all of its different formulations.

Another criterion of confirmation which is reflected in many methodological discussions and which claims a great deal of plausibility is called 'Prediction criterion.' Its basic idea is very simple; general hypothesis in science as well as in everyday usage are intended to enable us to anticipate future events; and hence it seems reasonable to count any prediction which is borne out by a subsequent operation as a confirming evidence.

But this too is not acceptable as a criterion of confirmation for the following reasons:

(a) Suppose we have a contradictory theory in which according to the rules of logic and statement and therefore any statement which can be verified to be true can be predicted. However, these true statements should not constitute a confirmation of the theory in any sense.

(b) Moreover, does the true prediction that 'bread is nourishing' confirm the theory T: (i) Bread is made of stone, and (ii) whatever is made of stone is nourishing?

C.T.K. Chari

LOGICAL ISSUES ABOUT THE CANONICAL FORMALISM IN CLASSICAL AND QUANTUM MECHANICS

H. R. Hanson,[5] not long ago, wrote: "What the practising physicist is likely to find difficult in many philosophical papers concerning the foundations of the quantum theory is a facile use of terms like 'realism', 'objectivism', 'subjectivism'." Undoubtedly a great deal of harm has been done by debating *ad nauseam* the alleged issues about "freedom *vs.* determinism" in the context of quantum physics. The usual talk about "waves" and "particles", however edifying in philosophical circles, is not scientifically enlightening. Philosophers of science trained in the rigours of modern science can perhaps atone for their metaphysical lapses by considering some of the severely technical yet highly significant issues in quantum mechanics [2], [3], [4]. I draw attention here to some logical questions about the canonical formalism in both classical and quantum mechanics.

A few general topological considerations may serve to guide the inquiry. The operation of forming the union of a finite number of sets may be denoted by $\cup$ and the union of an infinite number of sets constituting a family F may conveniently be denoted by $\underset{X\varepsilon F}{\Sigma} X$. It is obvious that the union of a family of subsets of R X Σ F is unaffected by the order of elements we choose from the family. The operation $\cup$ is unrestrictedly commutative in the usual topological treatment: $X\cup Y=Y\cup X$.

If a family F of subsets of R can be exhibited as a combination of two or more families G, H, ... of subsets of R, we can show that

$$\underset{X\varepsilon F}{\Sigma} X = \underset{X\varepsilon G}{\Sigma} X U \underset{X\varepsilon H}{\Sigma} U \ldots$$

In other words, the operation $\cup$, in the usual topological treatment, is unrestrictedly associative. If the topological intersection of sets is denoted by $\cap$, we can state that the operations of union and intersection distribute each other without restriction.

(i) $$X \cap \underset{Y\varepsilon F}{\Sigma} Y = \underset{Y\varepsilon F}{\Sigma} (X\cap Y).$$

(ii) $$X \cup \underset{Y\varepsilon F}{\pi} Y = \underset{Y\varepsilon F}{\pi}(X U Y).$$

If the set-complement X' of any subset X of R is defined to be the set of all elements of R which are *not* elements of X, then

(iii) $X\cap X'=0$;

(iv) $XUX'=1$.

The first equality states that there is no element which is both a member and not a member of X while the second equality states that any element is either a member or not a member of the set X. These two properties of the set-complement express the time-honoured principles of contradiction and excluded middle.

If set-union $\cup$, set-intersection $\cap$, and set-complement (′) are identified with the three Boolean operations of $+$, . , and (′), the algebra B_R of all subsets of R can be expounded as an instance of an abstract Boolean algebra calling for *either* the general distributive law *or* the existence of an atomic basis. Tarski proved that a complete Boolean algebra B in which the general distributive law is valid has necessarily an atomic basis.

Applying the general distributive law to the product, we have $\pi(\alpha+\alpha'),\ \alpha\varepsilon B,$

$$1=\pi(\alpha+\alpha')=\Sigma\, P(g),$$

where P(g) is the product of a set gCB such that, if α is any element of B, either $\alpha \varepsilon g$. or $\alpha' \varepsilon g$. Suppose that $t < P(g)$, where $<$ is a reflexive and transitive relation; then if $t \varepsilon g$, $P(g) < t$ and therefore, $t = P(g)$; and if $t' \varepsilon g$, then $t = t.\ P(g) = 0$. Hence $P(g)$ is an atom if it is not 0. Since $1 = \Sigma\, P(g)$ the non-zero products $P(g)$ —which must exist—constitute an atomic basis.

A Boolean set-ring may *generally* be defined as a family of sets which contains besides the sets, A, B, their union $A \cup B$, and their intersection, $A \cup B$. The difference, $A - B$, of two sets may be replaced conveniently by the *symmetric difference* defined by

$$A \oplus B = (A - B) \cup (B - A) = (A \cap B') \cup (B \cap A').$$

The binary operation $\oplus$ is only commutative and associative but is characterized by a group-operation not possessed by $\cap$ or $\cup$
as such. The properties

(1) $A \oplus O = A$;

(2) $A \oplus A = O$; and if $A \oplus X = O$, X must be equal to A, ensure that the family B_R of all subsets of a set R forms a *group* with respect to the symmetric difference. If $\oplus$ is commutative as well, as in the instance here considered, the group is abelian.

I turn from the topological preliminaries to the canonical formalism of classical mechanics. Suppose T is the kinetic energy and V is the potential energy of classical mechanics, then the Lagrangian function for a conservative holonomic system, with $L = (T - V)$, may be written

$$\frac{d}{dt}\,\frac{\partial L}{\dot{q}_k} - \frac{\partial L}{\partial q_k} = 0,$$

where K passes from 1 to n. The Lagrangian can be solved in generalized coordinates and associated momenta:

$$p_k = \frac{\partial L}{\partial q_k};\ \dot{p}_k = -\frac{\partial L}{\partial q_k}.$$

Introducing a function,

$$H = \sum_{k=1}^{n} p_k\, \dot{q}_k - L\,(q, \dot{q}),$$

we get

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} = -\dot{p}_k.$$

The Hamiltonian, i.e. the function H (p, q), was anticipated by Poisson and Lagrange. Where we have n Lagrangian equations (one for each degree of freedom) of the second order, we have 2n Hamiltonian equations of the first order.[9] In each case, the total number of arbitrary constants of integration which must be fixed by the boundary conditions is 2n. G.W. Mackey[7] has pertinently observed that the logic of classical mechanics may be exhibited as a Boolean algebra of all Boolean subsets of phase space.

The canonical equations may be expressed by using Poisson brackets. Let us consider *any* two continuous and differentiable functions of q_i, p_j, say g (q_i, p_j) and h (q_j, p_j). We write

$$(g, h) = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial p_j}\,\frac{\partial h}{\partial q_j} - \frac{\partial g}{\partial q_j}\,\frac{\partial h}{\partial p_j} \right)$$

and call the expression the Poisson bracket for g and h. The Poisson bracket has obvious utility. Suppose that $g = q_j$ and $h = q_k$; then for any j, k (1......n), we have

$$(q_j, q_k) = 0.$$

Similarly

$$(p_j, p_k) = 0.$$

and if $j \neq k$,

$$(p_j, q_k) = 0.$$

On the other hand. if $g = p_k$ and $h = q_k$,

$$(p_k, q_k) = 1;\ (q_k, p_k) = -1.$$

Functions of the p's and q's the Poisson bracket of which vanishes, are said to *commute* while those for which the Poisson bracket is equal to 1 are said to be *canonically conjugate*. Speaking generally, any function of the p's and q's the time-derivative of which is 1 will be canonically conjugate to H. Poisson brackets remain invariant under a canonical transformation, i.e. one which leaves the canonical equations unaltered. They have some algebraic properties of interest, e.g.

$$(g, h) = -(h, g);$$
$$\text{and } (g, c) = 0,$$

if C is a constant.

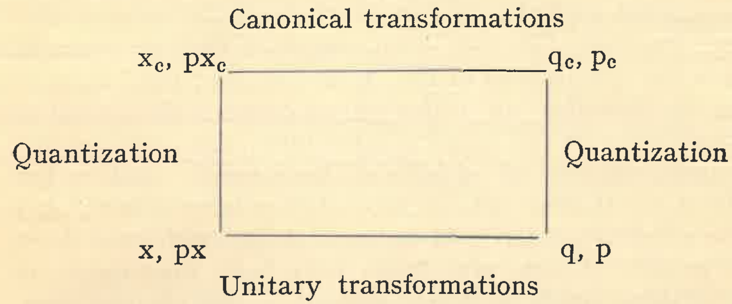
The transition from classical mechanics to quantum mechanics in textbooks[11] is usually effected by replacing the canonical variables, q_k, p_k, by Hermitian operators satisfying the commutative relations:

$$[q_k, q_k'] = [p_k, p_k'] = 0; \; [p_k, q_k'] = h_1 \; \delta_{kk'}.$$

The mechanical properties of the system are determined by its Hamiltonian formally taken over from classical theory, but reinterpreted as a Hermitian operator. In the same way, we may consider the properties of a quantized field as determined by its Hamiltonian $H = \int dxH$ or by its Lagrangian $L = \int dxL$, where H and L (the integrands) are the "differential Hamiltonian function" and the "differential Lagrangian function" respectively[11]. The classical field can be determined by the condition that the integral shall be stationary ($\delta I = 0$) for arbitrary variations $\delta \psi \sigma$ (x, t) and for an arbitrary choice of the integration region. The conditions, however, which must be satisfied by the extremals in the n-dimensional space (namely the Euler-Lagrange equations) are strong enough to ensure that $\delta I = 0$ for curves in the 2n-dimensional space (Hamilton's canonical equations), the end-points being, of course, fixed[9].

The classical Hamiltonian, $H = T + V$, is a function of generalized coordinates and momenta; $\dot{H} = 0$ or H is a constant. The quantum-mechanical Hamiltonian H in Schrödinger's fundamental equation, $H\psi = E\psi$, is an energy-operator. If x_c, px_c represent Cartesian coordinates, and q_c, p_c the canonical quantities, E. Merzbacher[8] suggests that we can represent the analogy between the

canonical transformations of classical mechanics and the unitary transformations of quantum mechanics thus:



It is quite possible to regard the field operator ψ (x) and the conjugate momenta π(x) of a field theory as a matrix representation of the abstract ring defined by the commutative rules:

$$[\psi(x), \psi(x')] = [\pi(x), \pi(x')] = 0;$$
$$[\psi(x), \pi(x')] = i\,\delta(x - x').$$

Two such representations are equivalent if they are related by a unitary transformation which must belong to the same Hilbert space in which the operators are defined. As Mackey remarks[7], if the logic of classical mechanics is exhibited by a Boolean algebra, the logic of quantum mechanics, in its orthodox expositions, may be represented as the logic of partially ordered sets of all closed subspaces of a separable infinite dimensional Hilbert space. The lattice of all subspaces of a vector space is an example of a partially ordered set which is not a Boolean algebra. It displays, however, many regularity properties.

The correspondence between the familiar differential operators and the self-adjoint operators of Hilbert space is, in general, complicated. J.R. Shewell[10] made a useful survey of the "quantization rules" formulated by Von Neumann, Weyl, Rivier, and Yvon, for associating real functions of the classical canonical quantities with Hermitian operators in a Hilbert space. T.F. Jordan[6] and E.C.G. Sudarshan and more recently J.P. Amiet have argued that, for each of these "quantization rules," the combination of two operators is not, in general, associated with the Poisson brackets of the corresponding function. It has been suggested that we confront here the breakdown of the "quantization rules" intended to establish an isomorphism between the canonical transformations of classical mechanics and the unitary transformations of quantum mechanics[3]. Difficulties about the Hilbert space representation have arisen[2,4] and the adequacy of the field-theoretical approach has been disputed[4]. There are those who hold that quantum mechanics needs a logic without a distributive law. Earlier in the paper I showed how the complete Boolean algebra in which the general distributive law is valid has necessarily an atomic basis. It is possible to construct logics without an atomic basis. The logic of quantum mechanics calls for further clarification by those philosophers of science interested in foundations.

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Appendix I

B.S. Bhatnagar

MATHEMATICAL LOGIC—A GLIMPSE

LOGICAL truths in traditional logic are figured as laws of identity, non-contradiction and excluded middle. They were taken as fundamental to logical inference. Every valid argument had to conform to them. They constituted the primitive base, the axiomatic foundations of a logical system. Inside the system, they were supreme. Outside, their validity rested on their mapping a very general pattern of things—every thing is what it is, a thing cannot be and be at one and the same time, a thing cannot possess two opposite attributes at one and the same time. This source of validity may be termed ontological in some vague sense. But this source, broadly speaking, is scientific in as much as it takes into consideration the pattern of things as they exist. This abstracted skeleton constitutes the semantic core of logic.

Laws of logic, in a sense, reflect the basic structural pattern of things. Laws of identity, non-contradiction and excluded middle emphasize the idea of a substance and its attributes and thereby of a class, diversifying itself into sub-classes, each contributing to a final and complete view of the class one begins with. The idea of number, though as primordial as that of a class, had not yet become part of the semantic background.

Aristotle's syllogistic can be interpreted as a logic of classes. This logic is in consonance with the spirit of geometry, a science of shape and form. There exist a number of geometric representations of syllogistic reason-

ing. Due to the lack of a symbol for zero, the Greeks could not develop decimal notation and their arithmetical achievements remained very low. They could not arithmetize logic nor logicise arithmetic. They, one should say, geometricized logic and logicized geometry. Arithmetic and arithmetisation is a wild game, having no fixed rules and having no end. Godel's incompleteness theorem is a significant pointer in this direction. Arithmetic was nurtured by the Hindus but as they lacked a logicized geometry of the type of the Greeks, they could develop neither arithmetic, beyond a certain level, nor logic. It is, of course, for the historian to find the source of such remarkable differences in the different culture-patterns of the two countries.

Aristotle's syllogistic in the middle ages remained sterile. Superficial distinctions and artificial rules were added; the science of arithmetic was still off the scene. The so-called Arabic notation—the decimal notation with zero—was being introduced into Europe through Spain by the Arab conquerors. It had still to incubate and flourish. Instead of enriching and standardizing the concept of class, rules of distribution of terms to check the validity of syllogisms were added. These rules fitted well in the context of the reducibility of every statement into one of the four forms, A,E, I and O,—a syntactic hazard already taken by Aristotle. The distinctions of moods and figures were exaggerated.

In the 19th century the doctrine of the quantification of the predicate, an off-shoot of the doctrine of distribution of terms, gave logic a spurt of illusory growth in the hands of De Morgan and Hamilton. Many spurious forms, over and above the four standard ones, were added. Some of them, though apparently independent, were proved to be logically equivalent. De Morgan wrote about himself and Hamitlon as fighting like a dog and a cat. It was in the context of this acrimonious controversy and a welter of misdirected energy that Boole started to write on logic. He published his book, *Mathematical Analysis of Logic,* in 1847. The same year, De Morgan's *Formal Logic* was also published. Boole was the first to emphasize operations between classes and to give exact rules for manipulating formulae couched in class symbolism. Boole is the father of the algebra of logic. It is called algebra because it has certain structural affinities with the algebra of real numbers. This algebra of logic differs from the other algebra in as much as its elements are idempotent with respect to the relation of conjunction and disjunction, without any coefficients and no exponents. Jevons and Venn carried on the Boolean tradition, the former with the help of the logical alphabet and the latter with the help of Venn diagrams. Boole succeeded in giving logic a clearcut denotative garb, the button-hole red rose being the null class as against the previous attempts either by giving rules of distribution, or by quantifying the predicate and increasing the number of standard logical forms. Boole did not frame exact rules of syntax although he presented logic as a calculus. Leibnitz had, much earlier, insisted on having a universal language and a calculus of reasoning in order to facilitate scientific enquiry. Boole had, however, a sound grasp of the subject matter which was to be dealt with through symbols, namely, classes within a fixed domain or *universe of discourse,* a phrase introduced by De Morgan. His viewing of the subject matter of logic from a denotative angle was not new but his genius seized this aspect in its full generality, clearing it, once for all, of any admixture of connotation. The denotative aspect of class has, since, found an accredited place in logical theory and has come to stay in its own right. Within its context, it affords a precise rendering of syllogistic arguments. This movement culminated in the three volume work of Schroder, the last of which appeared in 1905. This algebra of logic is properly named Boole—Schroder algebra. Peirce and De Morgan had made significant contributions in the field of algebra of relations.

In this long process of clarification and codification, the central idea of set theory, the membership relation, had not emerged. Connotation and denotation were taken to constitute the whole milieu of logical discourse, connotation predominating over denotation. It was in this spirit that Mill wrote about the primacy of connotation over denotation. Johnson in his *Logic* (1916) suggests that the circumference of a circle which represents a class term may be taken as connotation as the shape and the existence of the inner area (denotation) depend on it. What about the ubiquitous null class? Johnson also proves the independent existence of the fourth figure by giving in support a syllogistic argument in the same figure. Keynes' *Studies and Exercises in Formal Logic* (revised edition, 1908), recognized by Johnson as final word on traditional logic, studies syllogism, in erudite detail, besides immediate inference and opposition. Johnson in his *Logic* summarizes the essence of syllogistic reasoning in two principles, the applicative and the implicative. The implicative is truth-functional, the transitive law of material implication and the applicative is UI (universal instantiation) of the quantification theory. This analysis of syllogistic reasoning is still a far cry from the modern neat derivation, utilizing natural deduction or axiomatic techniques.

The next step in the development comes through set theory. The credit for founding set theory as an independent discipline goes to Cantor. He developed it in an intuitive manner. One of the basic principles is the principle of abstraction, namely, that every propositional function gives rise to a class. The other two are the principle of extensionality and the principle of choice. With these new concepts, Dedekind was able to give a better account of the nature of continuity in the continuum of real numbers and also of the concept of number itself. His two essays on these two subjects are still outstanding. He showed that these two concepts could be dealt with arithmetically. Cantor's idea of a cardinal number as equipollence between sets and Dedekind's definition of an infinite set as a set which can be mapped on to its subset, paved the way for Frege's logicizing the concept of number. Cantor's so-called diagonal method, basically a device for establishing one-one correspondence between various sets of numbers and for determining their cardinality, showed that rational numbers, in spite of their discreteness and also algebraic numbers, though they contain rational numbers as a subset, are denumerable. The by-product of this proof was the proof of the existence of transcendental numbers though the proof was dubbed by the intuitionists as non-constructive.

Imagination and reasoning, so nicely combined, produced a veritable host of transfinite numbers, each possessing a higher cardinality than the previous one. At root is the simple trick of mapping; the power-set cannot be put into one-one correspondence with the original set. The continuum hypothesis and the generalized continuum hypothesis and the whole arithmetic of the transfinite cardinals which resulted from this line of research are still live subjects. Set theory and arithmetic cross and recross in a multiplicity of patterns and structures, enriching logic and mathematics in hundred and one ways, opening a vast field of research in both. The idea of completed or actual infinite, introduced by Cantor, though disapproved by Gauss, was accepted by Hilbert.

Peano and his school were working primarily on two problems: firstly, they wanted to give mathematical reasoning a better symbolic form and secondly, as a result, to analyse mathematical reasoning itself i.e. to study its logic. Peano is also famous for the postulates he gave for arithmetizing mathematics. The whole of mathematics was derived from a few undefined notions—number, zero, successor—and five postulates. He applied axiomatic techniques to arithmetic.

In contrast, Frege was interested in giving an account

of these fundamental arithmetical ideas in terms of the notion of a set. He wanted, as is said, to logicize arithmetic. His ideas, however, were not given due recognition at that time. Russell, later on, salvaged his work and incorporated his ideas in his *Principles of Mathematics* (1903). Frege's *Begriffs Schrift* (1879) has been termed by Quine as the first real work on modern logic. In his opinion, by emphasizing the role of the individual variable in quantification theory, it is Frege and not Boole who is the real founder of modern logic. Propositional function replaces the old idea of a predicate; 'man' becomes 'X is human.' The individual variable in 'X is human' may be instantiated or generalized and in both cases it gives rise to a proposition. Predicates are classified according to the number of the individual variables they contain; one-place, two-place,.... In 'a is equal to b' the subject is the ordered pair $\langle$ a, b $\rangle$. By adopting this analysis and notation logic achieved a new level of intellection. Symbolism in terms of classes could be defined in terms of quantification.

Frege in his *Die Grundlagan der Arithmetik*, the principal work in which the thesis of logicizing arithmetic has been carried out, had utilized the principle of abstraction—borrowed from Cantor—as one of his axioms. Russell showed (1903) by using the matrix, 'X is not a member of X' that the above principle led to a contradiction if applied to the set of all sets. This is known as Russell's paradox. Frege saw an undoing of his whole achievement in this. Set theory seemed doomed. Russell and Zermelo both suggested (1908) ways to circumvent the paradox. Zermelo suggested the imposition of a new condition on the set. His axiom is called the axiom of separation. Set theory got a new direction. Fraenkel, Neumann, Skolem, Godel, Bernays have given it its present all-embracing character. Zermelo's repair was in the true spirit of logic and arithmetic.

Russell, on the other hand, transcended this contradiction by formulating his theory of Types. His theory raised ontological as well as mathematical issues, some of which were extraneous and irrelevant to the arithmetical tradition. *Principia Mathematica*, as a result, has become a classic. It is in the logistic tradition; from a few fundamental logical concepts it derives the whole of logic and mathematics. From unanalysed proposition it goes to propositional functions, to classes, to relations, to functions, to number and finally to cardinal and ordinal arithmetic. The promised volume on geometry was never published. The thesis—reduction of mathematics to logic—is known in literature as logicism. Quine's *Mathematical Logic* and Rosser's *Logic for Mathematicians* are both in this tradition.

It may be proper, at this stage, to take a glance at the axiomatic technique as it had developed by the beginning of the 20th century. In the 19th century it was used in geometry to prove (unsuccessfully) the dependence of the parallel postulate. As a result of this unsuccessful attempt, non-Euclidean geometries were discovered. It got established that in any chain of proof in geometry, it is logic that counts and not geometrical terms. The axiomatic method tended to become more and more abstract. It was employed in its abstract form with great success in geometry, Euclidean, non-Euclidean and projective by Hilbert, Pieri, Klein, Veblen and others. In 1904 Huntington published his set of postulates for the abstract Boolean algebra; within the deductive system, symbols were to be treated as mere marks with no connotative reverberations. Precise formal rules guided derivations. With this emphasis on formal structure, the questions of consistency, completeness and independence of the postulates became relevant. Hilbert had already raised these questions in his *Foundations of Geometry* (1899) and answered them in the then prevalent method of finding a model. His model was arithmetic. Huntington adapted this technique in *Continuum and the Types of Order* which gives sets of postulates for various types of order (quasi, simple,

discrete) and also for continuity and discusses their consistency and independence through models. In consonance with this movement, Sheffer and his school define logic as the science of pure form or structure. This view of logical truth does tend to conjure up in our minds images of cold, statue-like majestic forms, existing in a rarefied vacuum, immutable and eternal. We may also mention in this connection that in developing logic as an abstract Boolean algebra, a difficulty is faced when it comes to formalizing propositional logic. Propositional logic can, of course, be got from abstract Boolean algebra through interpretation; it is its 2-valued form. But to give it the form of an independent abstract system of logic, logic itself has to be assumed in the form of rules which are to govern derivations. Thus, there is logocentric predicament. Logic has to be assumed to prove logic. It is only in later developments when logic is developed through logistic techniques that this predicament is resolved. Logic was again in danger of slipping off its arithmetic-set-theoretical foothold and identifying itself with a sheer wealth of symbolism. 'Truth is never pure and simple' says Oscar Wilde. Sussane K. Langer's *Symbolic Logic* (1936) is in this tradition.

Contemporaneously Lewis was working on his system of strict implication as against the system of material implication as worked out in *Principia Mathematica*. He thought systems of logic based on and interpreted through connotation in contradistinction to denotation as found in Boolean algebra, would give a far better account of logical discourse. Syllogistic in the hands of Aristotle was ambiguous on this point. A purely denotative or connotative account of syllogism does not validate the nine Aristotelian syllogisms—four strengthened and five weakened. He supported the connotative viewpoint, thinking it to be more natural and philosophic. Psychological and philosophical considerations do not touch the hard core of mathematical thought; they may linger on and delineate its peripheral trappings only. The system of strict implication turned out to be modal in character. It had cut itself off from the main stream of logical tradition as it was unfolding itself in the early 20th century.

The torch-bearer of the true mathematical tradition, Hilbert, himself a creative mathematician of very high rank, did not swerve from the straight, though rocky and bumpy, arithmetical road. His contributions to mathematics and mathematical logic are varied and far-reaching. He had blessed Cantor and his creatures when the whole mathematical world seemed to be up against him. Hilbert fought against intuitionists who were out to discredit large slices of classical mathematics. He gave rise to a new science, the science of meta-mathematics. Meta-mathematics or meta-theory, as it is called in logical contexts, studies an important aspect of formal systems.

Logistic technique as already doveloped in *Principia Mathematica* was refined still more. A precise statement of primitive vocabulary, defined and undefined, notion of a *wff* (well formed formula) through recursive definition, notion of proof which is to be effective so that it may be capable of being checked, rules governing derivations, become the essentials of a formal system. Underlying it there is an intended interpretation. Whatever logic there is, it is part of the system. Nothing is taken for granted; its logic—intuitionistic or otherwise—has to evolve from within. The rules which sanction derivations do not explicate the concept of logical implication or validity because if they, in any way, define logical implication, the formal system will be vitiated by the fallacy of *petitio principii*. The rules have to fulfil, on the other hand, two criteria of soundness and completeness. The rules are said to be sound if they transmit the truth of the premises to the theorems derived from them; they are said to be complete, if they enable us to derive all the theorems of the subject matter under

study. In this study, a clear-cut distinction has to be made between the object language—the language of the formal system—and the meta-language—the language through which it is to be studied. This distinction was first emphasized by Frege. The rules and definition of a wff are given in meta-language.

As in the case of abstract axiom systems, the problem of consistency, of independence, and of completeness of the axioms of a formal system—now the logistic one—arise in a new context. These problems of consistency, completeness and independence along with the problem of decidability or undecidability constitute the special subject matter of meta-theory. The proofs have to be syntactic and should strictly remain within the four walls of basic symbolism, There has to be no reference to interpretation or model. The technique permitted to solve these problems has to be finitary. The principal tool is mathematical induction. Matrix tables are given for the relations within the system, This process reduces relations within the logistic system to relations of elementary arithmetic. The main aim behind all these techniqnes is to expose the full potentiality of symbolic structures with caution, care and sense. They keep a check on unrestricted generalizations and reduce to a minimum the chances of the emergence of a paradox. With all this emphasis on a purely syntactic approach, Hilbert had a firm grasp of the fundamental arithmetical set matrix, the semantic canvas as it may be called, out of which mathematical disciplines are born and fed.

Godel's contributions to meta-theory and logic are very profound. His incompleteness theorem which utilizes the technique of arithmetization of syntax, though shattered for all time the hopes of the formalist to establish consistency and completeness through meta-theoretical methods, has focussed discussion on proof procedures. He has shown that in elementary number theory proof procedures cannot be complete and addition of new ad hoc axioms will not remedy the situation.

The apparent never ending luxuriance of symbols, created through recursive definitions, will not match the inherent vitality and richness of arithmetic-set theoretical context which gives them validity and significance. To name only one instance, Boolean algebras have an almost embarrassingly rich structure. The simplest example is the field of integrers modulo 2. For quantification theory and mathematics the universe of discourse—the non-empty universe—is the set of natural numbers. This set is very rich in its characteristics. Being basic to every aspect of creation, it has permeated inextricably into the fabric of every other aspect of thinking.

No wonder, mathematics has been termed as the queen of sciences. Physics, which studies general structure of matter, often finds its concreteness melting into mathematical equations. A major portion of physics can be given the form of the theory of groups which may occupy only a few pages. Thus, mathematics and logic study the very general structure of the material environment. The ideas of number, set, all, some, true and false are as old as man. Quine says that logical connectives, quantifiers and the idea of class membership suffice to express mathematical or perhaps all discourse. In a way, logic and mathematics arise from concrete experience and revert to it for correction and guidance. Through their applications both become chastened and help to create a base for science and technology. Boolean algebra as an instrument for calculation expedited the manufacture of atom bomb by about six months. Logic in its Boolean form has applications in electrical circuits. Time is on the side of logic and mathematics and imagination can hardly encompass their future.

Broadly conceived, logic is very close to physical science. The old Aristotalian context of attribute and substance, partly philosophical and partly grammatical, which was the source of validity for the various laws of

logic, has given place to a clear statement of the conditions of validity of logical formulas, e.g. in the case of first order predicate calculus it runs as follows:—

> A schema is valid if, and only if, for every choice of a non-empty universe U, it comes out true under all interpretations within U of its predicate letters; i.e. true for all sub-classes of U as extensions of its predicate letters. It can be extended to predicates occurring polyadically. Further extensions to cover sentence letters and free variables can also be effected. The validity of the propositional logic is also derivable from this. Thus mathematical logic has as much of objective content to suppoct its symbolic structures as any other physical sciences. There is one added advantage; this base is shared by every science. Logic is, par excellence, the science of sciences.

In this context, the three philosophical theories of mathematics, logicism, formalism and intuitionism, do not exclude each other but rather are seen to lay emphasis on different aspects of a concrete and complex background. Logicism, in the Frege—Russell form, takes the notion of set as fundamental and defines number and number—theoretic concepts in its terms. It thus primarily concerns itself with the basic semantic background, the set-number matrix. It reduces the base to a monolithic structure—the structure of sets. So it tends to become philosophical rather than empirical and scientific. It seems to have closed the mathematical enquiry at one end—a feat most probably unattainable in an expanding universe with a fair sprinkling of acute minds.

The formalist, on the other hand, is chiefly interested in the exploration and mapping out of this matrix. He knows symbols are his best and the only tools. The concepts they stand for are sometimes so tenuous that they hardly suffice to keep a track of them. Yet he knows that symbols are the essence of mathematics. To reap the best of both the worlds, he has created the new science of meta-theory whose main purpose is to experiment with symbolic structures as divorced from any semantic background, often with the Godel technique of arithmetization and to ascertain what inferences can be drawn regarding the objective arithmetical-set matrix which they purport to represent. But first and last, in spite of his excursions into meta-theory, a formalist is a working mathematician, intent on unravelling the intricacies of the basic arithmetical-set context. Like a working craftsman, he has no fixed techniques, no set boundaries to his vast stores of imagination, intuition, conjecture and reasoning. He lives and thrives on experimentation. He is a born optimist and pragmatist. He may confuse ends and means, contradiction and truth, in his mad search after truth. Emerson's well-known dictum—consistency is the hobgoblin of little minds—suits him.

Intuitionism, in contrast, is tame and sober. It always tries to remain in touch with the semantic background, the set of natural numbers. For it, the background is statue-like, fixed and pedestalled. Its attitude toward it is reverential and mystic. It does sense here a rich variety and complexity of structure. But to study and interpret it, it recommends a cautious, finitary and constructive approach. Even the indirect method of proof, as hoary as Euclid, does not possess the proper rectitude. Indirect proof, it asserts, does not help us to construct a mathematical entity whose existence it enables us to prove. It seems to believe in some sort of an upper bound in the use of logic and symbols in mathematics. It is perhaps forgotten that logic and symbols are the two chief implements of mathematics and if initial restrictions are placed on their use, the very spirit of mathematics may well be choked. No science, much less mathematics, will accept a strait jacket. Science has grown with democracy, and it is only the democratic spirit which can keep it alive and vigorous.

Appendix II

N.D. Gautam

THREE THEOREMS OF GODEL —A SURVEY

MATHEMATICS is a mansion in the making. Man began to build it as he learnt to count and to draw lines. Construction work has been going on since. By the end of the nineteenth century, the superstructure became so big that the foundation began to give way: Cantor built the theory of sets and transfinite numbers, which ended in the Burali Forti paradox: there exists a cardinal number greater than the greatest cardinal number. Mathematicians became cautious—even suspicious. The foundation appeared to sink, the superstructure appeared to fall. They knew that logic was at the foundation of mathematics. They began to examine it critically. Some started building set theory on the tried axiomatic method. Zermelo axiomatised set theory in *"Untersuchungen ueber die Grundlagen der Mengenlehre"*[1] Almost simultaneously, in 1925, appeared the two axiom systems of Fraenkel[2] and of Von Neumann[3]. Two more systems will be mentioned in the sequel.

Frege had, perhaps, the premonition that the foundations of mathematics would give way. He tried to put

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1. *Math. Ann*, Vol. 65 pp. 261-48-1908.
2. Untersuchungen ueber die Grundlagen der Mengenlehre. *Math. Ztschr*, Vol. 22 pp. 250-273-1925.
3. Fine Axiomatisierung der Mengenlehre *J.F. Mcth.* Vcl. 154, pp. 219-240-1950.

it on a sound foundation in *Foundations of Arithmetic.*[1] Just then Russell discovered the paradox of the class of all those classes which are not members of themselves, giving a great shock to Frege. In turn, Russell, in collaboration with Whitehead, examined the complete superstructure of mathematics from the foundation upwards in the monumental '*Principia Mathematica.*' They set up a theory of types, dividing mathematical entities into entities of the first type (individuals), entities of the second type (properties), entities of the third type (properties of properties) and so on.

The completeness theorem

More mathematicians probed into mathematical theories and the axiomatic method in general. A mathematical theory or a calculus starts with a certain number of *primitive words or signs.* Some of these signs are *specific* to the theory under study. Others are signs common to all theories. These may be divied into *variables* (Sometimes different kinds of variables for different types, such as $x, y, z, \ldots, X, Y, Z, \ldots$) and *logical signs*: propositional connectives—∼(not) & (and), *V* (or), →(if-then), ←→(if and only if); quantifiers—(X) (for all X), (ΣX) (there is an X such that). *Expressions* are constructed as finite sequences of signs. Some expressions are distinguished as *statements* some as *terms*. *Axioms* are laid down and rules are given to derive new statements from a given set of statements. The notion of *theorem* is defined recursively as an axiom or a statement which can be derived from a set of theorems by finite number of application of the rules.

In terms of the fundamental *syntactical* notion of theorem we define other syntactical concepts. A theory is *contradictory* if every statement is a theorem of the theory. If there is a statement, which is not a theorem, then the theory is *consistent*.

1. Traslated from German by J.L. Austin, Oxford, 1950.

In addition to these syntactical notions, semantical notions such as *true, valid, model,* etc., are also used in mathematics. The notions of truth and model for a formalised theory were analysed by Tarski in *The Concept of Truth in Formalised Languages.*[1] Like the notion of theorem, the notion of model for a theory is defined recursively. The existence of a universe of individuals, properties and attributes, etc. is postulated. An *interpretation is defined as a* mapping of the set of variables into the universe. Take, for example, the statement p(x), where x is a variable for individuals and p a variable for properties. An interpretation I is a *model* for p(x) if the individual I(x) possesses the property I(p). It is in this sense, when we say that the integers form a model of the group axioms. A statement is said to be *true* if every interpretation of it is a model for it.

Now a question arises: Under what conditions can we say that a true statement is a theorem and *vice versa*? This question was answered by Godel for what is called the *first order functional calculus.* A calculus is called first order functional calculus, if quantification is applied to the variables for individuals only. In his completeness theorem Godel established that a statement is true if and only if it is a theorem.[2] A very instructive proof of this theorem was given by Henkin.[3]

The Incompleteness Theorem

When we take up the formalisation of mathematics, we find that there are parts of mathematics which cannot be adequately formalised in the first order functional calculus. Quantification of variables for properties be-

1. Included in his *Logic, Semantics and Matamathematics*, Oxford, 1956, pp. 152—278.

2. Die Vollstaendigkeit der Axiome des logischen Funktionenkalkuels, *Monaatshefte fur Math. Und. Phys.*, Vol. 37—1930.

3. *Journal of Symbolic Logic*, Vol. 14—1949.

comes unavoidable. A calculus in which variables other than those for individuals can be quantified is called a *higher order functional calculus*. Is it possible to lay down the axioms and rules of inference of such a calculus once for all, so that all true statements of it can be proved as theorems? Godel showed that no such calculus which contains the usual arithmetic is possible. Whatever be the axioms and whatever be the rules of inference, it is always possible to produce true statements which are not theorems. It is, of course, assumed that such a calculus is consistent.[1] He also established the impossibility of proving the consistency of any such calculus by finitary methods.

Hasenjaeger proved the incompleteness theorem of the higher order functional calculus by using the theorem on the *undecidability* of the first order functional calculus. A calculus or a theory is called *decidable* if there exists an *effective method* which decides in a finite number of steps whether a given statement is a theorem or not. In principle, it should be possible to construct a machine to carry out the instructions about such an effective method. Mathematicians have always been in search of such effective methods or algorithms. There exists, for example, an effective method for finding the square root of a given number. It was discovered by Aryabhatta. This vague intuitive notion of effective method was analysed into the clearly defined notion of λ*-definability* by Church.

Turing showed that λ*-definability* is equivalent to his notion of *computability*.[2] Finally Kleene established

1. Godel: Ueber formal unentscheidbare Saetze der *Principia Mathematica* and verwandter Systeme, *Monatsh. Math. Phy.*, Vol. 38, pp. 173-198 —1931. An unsolvable problem of elementary number theory, American Journal of Maths., Vol. 58, 1936, translated into English with an introduction by Braithite *On Formally Undecidable in the Principia Mathematica*, 1964.

2. On computable numbers, *Proc. London. Math. Sec.*, Ser. 2, Vol. 42, 1937.

the equivalence of the two notions of λ*-definability* and *recursiveness*.[1]

The Consistency Theorem

As paradoxes emerged, some mathematicians began to fear that mathematics may be a house built on sand. The problem of consistency became urgent. Hilbert and his followers devoted much attention and effort to it. They wanted to establish the consistency of mathematics by finite means, which could not be done.[2]

Godel established the consistency of the axiom of choice and of the generalised continuum hypothesis with the axioms of set theory.[3] He proved that the axiom of choice and Cantor's generalised continuum hypothesis (i.e. the proposition that $2^{Na} = N_a + 1$ for any ordinal number a, Na being the corresponding transfinite cardinal) are consistent with the other axioms of set theory if these axiom are consistent. His system of axioms for set theory does not, of course, include the axiom of choice and the generalised continuum hypothesis. The system is essentially due to Bernays.[4]

1. —definability and recursiveness, *Duke Math. J.*, Vol. 2, 1936,

2. An account of Hilbert's Beweistheorie is given in Hilbert and Bernays: *Grundlagen der Mathematik*, Vol, I, 1934, Vol. II, 1939.

3. The consistency of the Axiom of Choice, *Ann. Math. Studies.* No. 3. Princeton, 1940.

4. *Journal of Symbolic Logic:* Vol. 2, pp. 65-77. Vol. 6, pp. 1-17; Vol. 7, pp. 65-89, 133-145; Vol. 8, pp. 89-106; Vol. 13, pp. 65-79; Vol. 19, pp. 81-96.

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