# CHAPTER I.

## DEFINITION OF PURE MATHEMATICS.

**1.** Pure Mathematics is the class of all propositions of the form "$p$ implies $q$," where $p$ and $q$ are propositions containing one or more variables, the same in the two propositions, and neither $p$ nor $q$ contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

to which, in the present work, an answer professing itself demonstrable with mathematical certainty will be given—an answer which, however, consists in reducing the above problems to problems in pure logic, which last will not be found satisfactorily solved in what follows.

3. The Philosophy of Mathematics has been hitherto as controversial, obscure and unprogressive as the other branches of philosophy. Although it was generally agreed that mathematics is in some sense true, philosophers disputed as to what mathematical propositions really meant: although something was true, no two people were agreed as to what it was that was true, and if something was known, no one knew what it was that was known. So long, however, as this was doubtful, it could hardly be said that any certain and exact knowledge was to be obtained in mathematics. We find, accordingly, that idealists have tended more and more to regard all mathematics as dealing with mere appearance, while empiricists have held everything mathematical to be approximation to some exact truth about which they had nothing to tell us. This state of things, it must be confessed, was thoroughly unsatisfactory. Philosophy asks of Mathematics: What does it mean? Mathematics in the past was unable to answer, and Philosophy answered by introducing the totally irrelevant notion of mind. But now Mathematics is able to answer, so far at least as to reduce the whole of its propositions to certain fundamental notions of logic. At this point, the discussion must be resumed by Philosophy. I shall endeavour to indicate what are the fundamental notions involved, to prove at length that no others occur in mathematics, and to point out briefly the philosophical difficulties involved in the analysis of these notions. A complete treatment of these difficulties would involve a treatise on Logic, which will not be found in the following pages.

4. There was, until very lately, a special difficulty in the principles of mathematics. It seemed plain that mathematics consists of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to existing mathematics. Not only the Aristotelian syllogistic theory, but also the modern doctrines of Symbolic Logic, were either theoretically inadequate to mathematical reasoning, or at any rate required such artificial forms of statement that they could not be practically applied. In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, *i.e.* the *à priori* knowledge of space and time. Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation. By the help of ten principles of deduction and ten other premisses of a general logical nature (*e.g.* "implication is a relation"), all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premisses.

In this statement, Mathematics includes not only Arithmetic and Analysis, but also Geometry, Euclidean and non-Euclidean, rational Dynamics, and an indefinite number of other studies still unborn or in their infancy. The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself.

**5.** The general doctrine that all mathematics is deduction by logical principles from logical principles was strongly advocated by Leibniz, who urged constantly that axioms ought to be proved and that all except a few fundamental notions ought to be defined. But owing partly to a faulty logic, partly to belief in the logical necessity of Euclidean Geometry, he was led into hopeless errors in the endeavour to carry out in detail a view which, in its general outline, is now known to be correct*. The actual propositions of Euclid, for example, do not follow from the principles of logic alone; and the perception of this fact led Kant to his innovations in the theory of knowledge. But since the growth of non-Euclidean Geometry, it has appeared that pure mathematics has no concern with the question whether the axioms and propositions of Euclid hold of actual space or not: this is a question for applied mathematics, to be decided, so far as any decision is possible, by experiment and observation. What pure mathematics asserts is merely that the Euclidean propositions follow from the Euclidean axioms—*i.e.* it asserts an implication: any space which has such and such properties has also such and such other properties. Thus, as dealt with in pure mathematics, the Euclidean and non-Euclidean Geometries are equally true: in each nothing is affirmed except implications. All propositions as to what actually exists, like the space we live in, belong to experimental or empirical science, not to mathematics; when they belong to applied mathematics, they arise from giving to one or more of the variables in a proposition of pure mathematics some constant value satisfying the hypothesis, and thus enabling us, for that value of the variable, actually to assert both hypothesis and consequent instead of asserting merely the implication. We assert always in mathematics that if a certain assertion $p$ is true of any entity $x$, or of any set of entities $x, y, z, \ldots$, then some other assertion $q$ is true of those entities; but we do not assert either $p$ or $q$ separately of our entities. We assert a relation between the assertions $p$ and $q$, which I shall call *formal implication*.

**6.** Mathematical propositions are not only characterized by the fact that they assert implications, but also by the fact that they contain *variables*. The notion of the variable is one of the most difficult with which Logic has to deal, and in the present work a satisfactory theory

* On this subject, cf. Couturat, *La Logique de Leibniz*, Paris, 1901.

as to its nature, in spite of much discussion, will hardly be found. For the present, I only wish to make it plain that there are variables in all mathematical propositions, even where at first sight they might seem to be absent. Elementary Arithmetic might be thought to form an exception: $1+1=2$ appears neither to contain variables nor to assert an implication. But as a matter of fact, as will be shown in Part II, the true meaning of this proposition is: "If $x$ is one and $y$ is one, and $x$ differs from $y$, then $x$ and $y$ are two." And this proposition both contains variables and asserts an implication. We shall find always, in all mathematical propositions, that the words *any* or *some* occur; and these words are the marks of a variable and a formal implication. Thus the above proposition may be expressed in the form: "Any unit and any other unit are two units." The typical proposition of mathematics is of the form "$\phi(x, y, z, \ldots)$ implies $\psi(x, y, z, \ldots)$, whatever values $x$, $y$, $z$, ... may have"; where $\phi(x, y, z, \ldots)$ and $\psi(x, y, z, \ldots)$, for every set of values of $x$, $y$, $z$, ..., are propositions. It is not asserted that $\phi$ is always true, nor yet that $\psi$ is always true, but merely that, in all cases, when $\phi$ is false as much as when $\phi$ is true, $\psi$ follows from it.

The distinction between a variable and a constant is somewhat obscured by mathematical usage. It is customary, for example, to speak of parameters as in some sense constants, but this is a usage which we shall have to reject. A constant is to be something absolutely definite, concerning which there is no ambiguity whatever. Thus 1, 2, 3, $e$, $\pi$, Socrates, are constants; and so are *man*, and the human race, past, present and future, considered collectively. Proposition, implication, class, etc. are constants; but a proposition, any proposition, some proposition, are not constants, for these phrases do not denote one definite object. And thus what are called parameters are simply variables. Take, for example, the equation $ax+by+c=0$, considered as the equation to a straight line in a plane. Here we say that $x$ and $y$ are variables, while $a$, $b$, $c$ are constants. But unless we are dealing with one absolutely particular line, say the line from a particular point in London to a particular point in Cambridge, our $a$, $b$, $c$ are not definite numbers, but stand for *any* numbers, and are thus also variables. And in Geometry nobody does deal with actual particular lines; we always discuss *any* line. The point is that we collect the various couples $x$, $y$ into classes of classes, each class being defined as those couples that have a certain fixed relation to one triad $(a, b, c)$. But from class to class, $a$, $b$, $c$ also vary, and are therefore properly variables.

7. It is customary in mathematics to regard our variables as restricted to certain classes: in Arithmetic, for instance, they are supposed to stand for numbers. But this only means that *if* they stand for numbers, they satisfy some formula, *i.e.* the hypothesis that they are numbers implies the formula. This, then, is what is really

asserted, and in this proposition it is no longer necessary that our variables should be numbers: the implication holds equally when they are not so. Thus, for example, the proposition "$x$ and $y$ are numbers implies $(x+y)^3 = x^3 + 2xy + y^3$" holds equally if for $x$ and $y$ we substitute Socrates and Plato*: both hypothesis and consequent, in this case, will be false, but the implication will still be true. Thus in every proposition of pure mathematics, when fully stated, the variables have an absolutely unrestricted field: any conceivable entity may be substituted for any one of our variables without impairing the truth of our proposition.

8. We can now understand why the constants in mathematics are to be restricted to logical constants in the sense defined above. The process of transforming constants in a proposition into variables leads to what is called generalization, and gives us, as it were, the formal essence of a proposition. Mathematics is interested exclusively in *types* of propositions; if a proposition $p$ containing only constants be proposed, and for a certain one of its terms we imagine others to be successively substituted, the result will in general be sometimes true and sometimes false. Thus, for example, we have "Socrates is a man"; here we turn Socrates into a variable, and consider "$x$ is a man." Some hypotheses as to $x$, for example, "$x$ is a Greek," insure the truth of "$x$ is a man"; thus "$x$ is a Greek" implies "$x$ is a man," and this holds for all values of $x$. But the statement is not one of pure mathematics, because it depends upon the particular nature of *Greek* and *man*. We may, however, vary these too, and obtain: If $a$ and $b$ are classes, and $a$ is contained in $b$, then "$x$ is an $a$" implies "$x$ is a $b$." Here at last we have a proposition of pure mathematics, containing three variables and the constants *class*, *contained in*, and those involved in the notion of formal implications with variables. So long as any term in our proposition can be turned into a variable, our proposition can be generalized; and so long as this is possible, it is the business of mathematics to do it. If there are several chains of deduction which differ only as to the meaning of the symbols, so that propositions symbolically identical become capable of several interpretations, the proper course, mathematically, is to form the class of meanings which may attach to the symbols, and to assert that the formula in question follows from the hypothesis that the symbols belong to the class in question. In this way, symbols which stood for constants become transformed into variables, and new constants are substituted, consisting of classes to which the old constants belong. Cases of such generalization are so frequent that many will occur at once to every mathematician, and innumerable instances will be given in the present work. Whenever two sets of terms have mutual relations of the same

* It is necessary to suppose arithmetical addition and multiplication defined (as may be easily done) so that the above formula remains significant when $x$ and $y$ are not numbers.

type, the same form of deduction will apply to both. For example, the mutual relations of points in a Euclidean plane are of the same type as those of the complex numbers; hence plane geometry, considered as a branch of pure mathematics, ought not to decide whether its variables are points or complex numbers or some other set of entities having the same type of mutual relations. Speaking generally, we ought to deal, in every branch of mathematics, with any class of entities whose mutual relations are of a specified type; thus the class, as well as the particular term considered, becomes a variable, and the only true constants are the types of relations and what they involve. Now a *type* of relation is to mean, in this discussion, a class of relations characterized by the above formal identity of the deductions possible in regard to the various members of the class; and hence a type of relations, as will appear more fully hereafter, if not already evident, is always a class definable in terms of logical constants*. We may therefore define a type of relations as a class of relations defined by some property definable in terms of logical constants alone.

**9.** Thus pure mathematics must contain no indefinables except logical constants, and consequently no premisses, or indemonstrable propositions, but such as are concerned exclusively with logical constants and with variables. It is precisely this that distinguishes pure from applied mathematics. In applied mathematics, results which have been shown by pure mathematics to follow from some hypothesis as to the variable are actually asserted of some constant satisfying the hypothesis in question. Thus terms which were variables become constant, and a new premiss is always required, namely: this particular entity satisfies the hypothesis in question. Thus for example Euclidean Geometry, as a branch of pure mathematics, consists wholly of propositions having the hypothesis "$S$ is a Euclidean space." If we go on to: "The space that exists is Euclidean," this enables us to assert of the space that exists the consequents of all the hypotheticals constituting Euclidean Geometry, where now the variable $S$ is replaced by the constant *actual space*. But by this step we pass from pure to applied mathematics.

**10.** The connection of mathematics with logic, according to the above account, is exceedingly close. The fact that all mathematical constants are logical constants, and that all the premisses of mathematics are concerned with these, gives, I believe, the precise statement of what philosophers have meant in asserting that mathematics is *à priori*. The fact is that, when once the apparatus of logic has been accepted, all mathematics necessarily follows. The logical constants themselves are to be defined only by enumeration, for they are so fundamental that all the properties by which the class of them might be defined presuppose

* One-one, many-one, transitive, symmetrical, are instances of types of relations with which we shall be often concerned.

From what has now been said, the reader will perceive that the present work has to fulfil two objects, first, to show that all mathematics follows from symbolic logic, and secondly to discover, as far as possible, what are the principles of symbolic logic itself. The first of these objects will be pursued in the following Parts, while the second belongs to Part I. And first of all, as a preliminary to a critical analysis, it will be necessary to give an outline of Symbolic Logic considered simply as a branch of mathematics. This will occupy the following chapter.

430. (5) The fifth argument seems to be designed to prove the Kantian apriority of space. There are, it says, necessary propositions concerning space, which show that the nature of space is not a "mere fact." We are intended to infer that space is an *à priori* intuition, and a psychological reason is given why we cannot imagine holes in space. The impossibility of holes is apparently what is called a necessity of thought. This argument again involves much purely logical discussion. Concerning necessities of thought, the Kantian theory seems to lead to the curious result that whatever we cannot help believing must be false. What we cannot help believing, in this case, is something as to the nature of space, not as to the nature of our minds. The explanation offered is, that there is no space outside our minds; whence it is to be inferred that our unavoidable beliefs about space are all mistaken. Moreover we only push one stage farther back the region of "mere fact," for the constitution of our minds remains still a mere fact.

The theory of necessity urged by Kant, and adopted here by Lotze, appears radically vicious. Everything is in a sense a mere fact. A proposition is said to be proved when it is deduced from premisses; but the premisses, ultimately, and the rule of inference, have to be simply assumed. Thus any ultimate premiss is, in a certain sense, a mere fact. On the other hand, there seems to be no true proposition of which there is any sense in saying that it might have been false. One might as well say that redness might have been a taste and not a colour. What is true, is true; what is false, is false; and concerning fundamentals, there is nothing more to be said. The only logical meaning of necessity seems to be derived from implication. A proposition is more or less necessary according as the class of propositions for which it is a premiss is greater or smaller*. In this sense the propositions of logic have the greatest necessity, and those of geometry have a high degree of necessity. But this sense of necessity yields no valid argument from our inability to imagine holes in space to the conclusion that there cannot really be any space at all except in our imaginations.

* Cf. G. E. Moore, "Necessity," *Mind*, N.S., No. 35.

# CHAPTER II.

## SYMBOLIC LOGIC.

**11.** Symbolic or Formal Logic—I shall use these terms as synonyms—is the study of the various general types of deduction. The word *symbolic* designates the subject by an accidental characteristic, for the employment of mathematical symbols, here as elsewhere, is merely a theoretically irrelevant convenience. The syllogism in all its figures belongs to Symbolic Logic, and would be the whole subject if all deduction were syllogistic, as the scholastic tradition supposed. It is from the recognition of asyllogistic inferences that modern Symbolic Logic, from Leibniz onward, has derived the motive to progress. Since the publication of Boole's *Laws of Thought* (1854), the subject has been pursued with a certain vigour, and has attained to a very considerable technical development*. Nevertheless, the subject achieved almost nothing of utility either to philosophy or to other branches of mathematics, until it was transformed by the new methods of Professor Peano†. Symbolic Logic has now become not only absolutely essential to every philosophical logician, but also necessary for the comprehension of mathematics generally, and even for the successful practice of certain branches of mathematics. How useful it is in practice can only be judged by those who have experienced the increase of power derived from acquiring it; its theoretical functions must be briefly set forth in the present chapter‡.

* By far the most complete account of the non-Peanesque methods will be found in the three volumes of Schröder, *Vorlesungen über die Algebra der Logik*, Leipzig, 1890, 1891, 1895.

† See *Formulaire de Mathématiques*, Turin, 1895, with subsequent editions in later years; also *Revue de Mathématiques*, Vol. VII, No. 1 (1900). The editions of the *Formulaire* will be quoted as *F.* 1895 and so on. The *Revue de Mathématiques*, which was originally the *Rivista di Matematica*, will be referred to as *R. d. M.*

‡ In what follows the main outlines are due to Professor Peano, except as regards relations; even in those cases where I depart from his views, the problems considered have been suggested to me by his works.

**12.** Symbolic Logic is essentially concerned with inference in general*, and is distinguished from various special branches of mathematics mainly by its generality. Neither mathematics nor symbolic logic will study such special relations as (say) temporal priority, but mathematics will deal explicitly with the class of relations possessing the formal properties of temporal priority—properties which are summed up in the notion of continuity†. And the formal properties of a relation may be defined as those that can be expressed in terms of logical constants, or again as those which, while they are preserved, permit our relation to be varied without invalidating any inference in which the said relation is regarded in the light of a variable. But symbolic logic, in the narrower sense which is convenient, will not investigate what inferences are possible in respect of continuous relations (*i.e.* relations generating continuous series); this investigation belongs to mathematics, but is still too special for symbolic logic. What symbolic logic does investigate is the general rules by which inferences are made, and it requires a classification of relations or propositions only in so far as these general rules introduce particular notions. The particular notions which appear in the propositions of symbolic logic, and all others definable in terms of these notions, are the logical constants. The number of indefinable logical constants is not great: it appears, in fact, to be eight or nine. These notions alone form the subject-matter of the whole of mathematics: no others, except such as are definable in terms of the original eight or nine, occur anywhere in Arithmetic, Geometry, or rational Dynamics. For the technical study of Symbolic Logic, it is convenient to take as a single indefinable the notion of a formal implication, *i.e.* of such propositions as "$x$ is a man implies $x$ is a mortal, for all values of $x$"—propositions whose general type is: "$\phi(x)$ implies $\psi(x)$ for all values of $x$," where $\phi(x)$, $\psi(x)$, for all values of $x$, are propositions. The analysis of this notion of formal implication belongs to the principles of the subject, but is not required for its formal development. In addition to this notion, we require as indefinables the following: Implication between propositions not containing variables, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and truth. By means of these notions, all the propositions of symbolic logic can be stated.

**13.** The subject of Symbolic Logic consists of three parts, the calculus of propositions, the calculus of classes, and the calculus of relations. Between the first two, there is, within limits, a certain parallelism, which arises as follows: In any symbolic expression, the

* I may as well say at once that I do not distinguish between inference and deduction. What is called induction appears to me to be either disguised deduction or a mere method of making plausible guesses

† See below, Part V, Chap. xxxvi.

letters may be interpreted as classes or as propositions, and the relation of inclusion in the one case may be replaced by that of formal implication in the other. Thus, for example, in the principle of the syllogism, if $a$, $b$, $c$ be classes, and $a$ is contained in $b$, $b$ in $c$, then $a$ is contained in $c$; but if $a$, $b$, $c$ be propositions, and $a$ implies $b$, $b$ implies $c$, then $a$ implies $c$. A great deal has been made of this duality, and in the later editions of the *Formulaire*, Peano appears to have sacrificed logical precision to its preservation *. But, as a matter of fact, there are many ways in which the calculus of propositions differs from that of classes. Consider, for example, the following: "If $p$, $q$, $r$ are propositions, and $p$ implies $q$ or $r$, then $p$ implies $q$ or $p$ implies $r$." This proposition is true; but its correlative is false, namely: "If $a$, $b$, $c$ are classes, and $a$ is contained in $b$ or $c$, then $a$ is contained in $b$ or $a$ is contained in $c$." For example, English people are all either men or women, but are not all men nor yet all women. The fact is that the duality holds for propositions asserting of a variable term that it belongs to a class, *i.e.* such propositions as "$x$ is a man," provided that the implication involved be formal, *i.e.* one which holds for all values of $x$. But "$x$ is a man" is itself not a proposition at all, being neither true nor false; and it is not with such entities that we are concerned in the propositional calculus, but with genuine propositions. To continue the above illustration: It is true that, for all values of $x$, "$x$ is a man or a woman" either implies "$x$ is a man" or implies "$x$ is a woman." But it is false that "$x$ is a man or woman" either implies "$x$ is a man" for all values of $x$, or implies "$x$ is a woman" for all values of $x$. Thus the implication involved, which is always one of the two, is not formal, since it does not hold for all values of $x$, being not always the same one of the two. The symbolic affinity of the propositional and the class logic is, in fact, something of a snare, and we have to decide which of the two we are to make fundamental. Mr McColl, in an important series of papers †, has contended for the view that implication and propositions are more fundamental than inclusion and classes; and in this opinion I agree with him. But he does not appear to me to realize adequately the distinction between genuine propositions and such as contain a real variable: thus he is led to speak of propositions as sometimes true and sometimes false, which of course is impossible with a genuine proposition. As the distinction involved is of very great importance, I shall dwell on it before proceeding further. A proposition, we may say, is anything that is true or that is

* On the points where the duality breaks down, cf. Schröder, *op. cit.*, Vol. II, Lecture 21.

† Cf. "The Calculus of Equivalent Statements," *Proceedings of the London Mathematical Society*, Vol. IX and subsequent volumes; "Symbolic Reasoning," *Mind*, Jan. 1880, Oct. 1897, and Jan. 1900; "La Logique Symbolique et ses Applications," *Bibliothèque du Congrès International de Philosophie*, Vol. III (Paris, 1901). I shall in future quote the proceedings of the above Congress by the title *Congrès*.

false. An expression such as "$x$ is a man" is therefore not a proposition, for it is neither true nor false. If we give to $x$ any constant value whatever, the expression becomes a proposition: it is thus as it were a schematic form standing for any one of a whole class of propositions. And when we say "$x$ is a man implies $x$ is a mortal for all values of $x$," we are not asserting a single implication, but a class of implications; we have now a genuine proposition, in which, though the letter $x$ appears, there is no real variable: the variable is absorbed in the same kind of way as the $x$ under the integral sign in a definite integral, so that the result is no longer a function of $x$. Peano distinguishes a variable which appears in this way as *apparent*, since the proposition does not depend upon the variable; whereas in "$x$ is a man" there are different propositions for different values of the variable, and the variable is what Peano calls *real**. I shall speak of propositions exclusively where there is no real variable: where there are one or more real variables, and for all values of the variables the expression involved is a proposition, I shall call the expression a *propositional function*. The study of genuine propositions is, in my opinion, more fundamental than that of classes; but the study of propositional functions appears to be strictly on a par with that of classes, and indeed scarcely distinguishable therefrom. Peano, like McColl, at first regarded propositions as more fundamental than classes, but he, even more definitely, considered propositional functions rather than propositions. From this criticism, Schröder is exempt: his second volume deals with genuine propositions, and points out their formal differences from classes.

### A. *The Propositional Calculus.*

**14.** The propositional calculus is characterized by the fact that all its propositions have as hypothesis and as consequent the assertion of a material implication. Usually, the hypothesis is of the form "$p$ implies $p$," etc., which (§ 16) is equivalent to the assertion that the letters which occur in the consequent are propositions. Thus the consequents consist of propositional functions which are true of all propositions. It is important to observe that, though the letters employed are symbols for variables, and the consequents are true when the variables are given values which are propositions, these values must be genuine propositions, not propositional functions. The hypothesis "$p$ is a proposition" is not satisfied if for $p$ we put "$x$ is a man," but it is satisfied if we put "Socrates is a man" or if we put "$x$ is a man implies $x$ is a mortal for all values of $x$."· Shortly, we may say that the propositions represented by single letters in this calculus are variables, but do not contain variables—in the case, that is to say, where the hypotheses of the propositions which the calculus asserts are satisfied.

* *F.* 1901, p. 2.

**15.** Our calculus studies the relation of *implication* between propositions. This relation must be distinguished from the relation of *formal* implication, which holds between propositional functions when the one implies the other for all values of the variable. Formal implication is also involved in this calculus, but is not explicitly studied: we do not consider propositional functions in general, but only certain definite propositional functions which occur in the propositions of our calculus. How far formal implication is definable in terms of implication simply, or material implication as it may be called, is a difficult question, which will be discussed in Chapter III. What the difference is between the two, an illustration will explain. The fifth proposition of Euclid follows from the fourth: if the fourth is true, so is the fifth, while if the fifth is false, so is the fourth. This is a case of material implication, for both propositions are absolute constants, not dependent for their meaning upon the assigning of a value to a variable. But each of them *states* a formal implication. The fourth states that if $x$ and $y$ be triangles fulfilling certain conditions, then $x$ and $y$ are triangles fulfilling certain other conditions, and that this implication holds for all values of $x$ and $y$; and the fifth states that if $x$ is an isosceles triangle, $x$ has the angles at the base equal. The formal implication involved in each of these two propositions is quite a different thing from the material implication holding between the propositions as wholes; both notions are required in the propositional calculus, but it is the study of material implication which specially distinguishes this subject, for formal implication occurs throughout the whole of mathematics.

It has been customary, in treatises on logic, to confound the two kinds of implication, and often to be really considering the formal kind where the material kind only was apparently involved. For example, when it is said that "Socrates is a man, therefore Socrates is a mortal," Socrates is *felt* as a variable: he is a type of humanity, and one feels that any other man would have done as well. If, instead of *therefore*, which implies the truth of hypothesis and consequent, we put "Socrates is a man implies Socrates is a mortal," it appears at once that we may substitute not only another man, but any other entity whatever, in the place of Socrates. Thus although what is explicitly stated, in such a case, is a material implication, what is meant is a formal implication; and some effort is needed to confine our imagination to material implication.

**16.** A definition of implication is quite impossible. If $p$ implies $q$, then if $p$ is true $q$ is true, *i.e.* $p$'s truth implies $q$'s truth; also if $q$ is false $p$ is false, *i.e.* $q$'s falsehood implies $p$'s falsehood*. Thus truth and falsehood give us merely new implications, not a definition of implication.

* The reader is recommended to observe that the main implications in these statements are formal, *i.e.* "$p$ implies $q$" *formally* implies "$p$'s truth implies $q$'s truth," while the subordinate implications are material.

If $p$ implies $q$, then both are false or both true, or $p$ is false and $q$ true; it is impossible to have $q$ false and $p$ true, and it is necessary to have $q$ true or $p$ false*. In fact, the assertion that $q$ is true or $p$ false turns out to be strictly equivalent to "$p$ implies $q$"; but as equivalence means mutual implication, this still leaves implication fundamental, and not definable in terms of disjunction. Disjunction, on the other hand, is definable in terms of implication, as we shall shortly see. It follows from the above equivalence that of any two propositions there must be one which implies the other, that false propositions imply all propositions, and true propositions are implied by all propositions. But these are results to be demonstrated; the premisses of our subject deal exclusively with rules of inference.

It may be observed that, although implication is indefinable, *proposition* can be defined. Every proposition implies itself, and whatever is not a proposition implies nothing. Hence to say "$p$ is a proposition" is equivalent to saying "$p$ implies $p$"; and this equivalence may be used to define propositions. As the mathematical sense of *definition* is widely different from that current among philosophers, it may be well to observe that, in the mathematical sense, a new propositional function is said to be defined when it is stated to be equivalent to (*i.e.* to imply and be implied by) a propositional function which has either been accepted as indefinable or has been defined in terms of indefinables. The definition of entities which are not propositional functions is derived from such as are in ways which will be explained in connection with classes and relations.

**17.** We require, then, in the propositional calculus, no indefinables except the two kinds of implication—remembering, however, that formal implication is a complex notion, whose analysis remains to be undertaken. As regards our two indefinables, we require certain indemonstrable propositions, which hitherto I have not succeeded in reducing to less than ten. Some indemonstrables there must be; and some propositions, such as the syllogism, must be of the number, since no demonstration is possible without them. But concerning others, it may be doubted whether they are indemonstrable or merely undemonstrated; and it should be observed that the method of supposing an axiom false, and deducing the consequences of this assumption, which has been found admirable in such cases as the axiom of parallels, is here not universally available. For all our axioms are principles of deduction; and if they are true, the consequences which appear to follow from the employment of an opposite principle will not really follow, so that arguments from the supposition of the falsity of an axiom are here subject to special fallacies. Thus the number of indemonstrable propositions may be capable of further reduction, and in regard to some of them I know of

* I may as well state once for all that the alternatives of a disjunction will never be considered as mutually exclusive unless expressly said to be so.

no grounds for regarding them as indemonstrable except that they have hitherto remained undemonstrated.

**18.** The ten axioms are the following. (1) If $p$ implies $q$, then $p$ implies $q$*; in other words, whatever $p$ and $q$ may be, "$p$ implies $q$" is a proposition. (2) If $p$ implies $q$, then $p$ implies $p$; in other words, whatever implies anything is a proposition. (3) If $p$ implies $q$, then $q$ implies $q$; in other words, whatever is implied by anything is a proposition. (4) A true hypothesis in an implication may be dropped, and the consequent asserted. This is a principle incapable of formal symbolic statement, and illustrating the essential limitations of formalism—a point to which I shall return at a later stage. Before proceeding further, it is desirable to define the joint assertion of two propositions, or what is called their logical product. This definition is highly artificial, and illustrates the great distinction between mathematical and philosophical definitions. It is as follows: If $p$ implies $p$, then, if $q$ implies $q$, $pq$ (the logical product of $p$ and $q$) means that if $p$ implies that $q$ implies $r$, then $r$ is true. In other words, if $p$ and $q$ are propositions, their joint assertion is equivalent to saying that every proposition is true which is such that the first implies that the second implies it. We cannot, with formal correctness, state our definition in this shorter form, for the hypothesis "$p$ and $q$ are propositions" is already the logical product of "$p$ is a proposition" and "$q$ is a proposition." We can now state the six main principles of inference, to each of which, owing to its importance, a name is to be given; of these all except the last will be found in Peano's accounts of the subject. (5) If $p$ implies $p$ and $q$ implies $q$, then $pq$ implies $p$. This is called *simplification*, and asserts merely that the joint assertion of two propositions implies the assertion of the first of the two. (6) If $p$ implies $q$ and $q$ implies $r$, then $p$ implies $r$. This will be called the *syllogism*. (7) If $q$ implies $q$ and $r$ implies $r$, and if $p$ implies that $q$ implies $r$, then $pq$ implies $r$. This is the principle of *importation*. In the hypothesis, we have a product of three propositions; but this can of course be defined by means of the product of two. The principle states that if $p$ implies that $q$ implies $r$, then $r$ follows from the joint assertion of $p$ and $q$. For example: "If I call on so-and-so, then if she is at home I shall be admitted" implies "If I call on so-and-so and she is at home, I shall be admitted." (8) If $p$ implies $p$ and $q$ implies $q$, then, if $pq$ implies $r$, then $p$ implies that $q$ implies $r$. This is the converse of the preceding principle, and is called *exportation*†. The previous illustration reversed will illustrate this principle. (9) If $p$ implies $q$ and $p$ implies $r$, then $p$ implies $qr$: in other words, a

* Note that the implications denoted by *if* and *then*, in these axioms, are formal, while those denoted by *implies* are material.

† (7) and (8) cannot (I think) be deduced from the definition of the logical product, because they are required for passing from "If $p$ is a proposition, then '$q$ is a proposition' implies etc." to "If $p$ and $q$ are propositions, then etc."

proposition which implies each of two propositions implies them both. This is called the principle of *composition*. (10) If $p$ implies $p$ and $q$ implies $q$, then "'$p$ implies $q$' implies $p$" implies $p$. This is called the principle of *reduction*; it has less self-evidence than the previous principles, but is equivalent to many propositions that are self-evident. I prefer it to these, because it is explicitly concerned, like its predecessors, with implication, and has the same kind of logical character as they have. If we remember that "$p$ implies $q$" is equivalent to "$q$ or not-$p$," we can easily convince ourselves that the above principle is true; for "'$p$ implies $q$' implies $p$" is equivalent to "$p$ or the denial of '$q$ or not-$p$,'" *i.e.* to "$p$ or '$p$ and not $q$,'" *i.e.* to $p$. But this way of persuading ourselves that the principle of reduction is true involves many logical principles which have not yet been demonstrated, and cannot be demonstrated except by reduction or some equivalent. The principle is especially useful in connection with negation. Without its help, by means of the first nine principles, we can prove the law of contradiction; we can prove, if $p$ and $q$ be propositions, that $p$ implies not-not-$p$; that "$p$ implies not-$q$" is equivalent to "$q$ implies not-$p$" and to not-$pq$; that "$p$ implies $q$" implies "not-$q$ implies not-$p$"; that $p$ implies that not-$p$ implies $p$; that not-$p$ is equivalent to "$p$ implies not-$p$"; and that "$p$ implies not-$q$" is equivalent to "not-not-$p$ implies not-$q$." But we cannot prove without reduction or some equivalent (so far at least as I have been able to discover) that $p$ or not-$p$ must be true (the law of excluded middle); that every proposition is equivalent to the negation of some other proposition; that not-not-$p$ implies $p$; that "not-$q$ implies not-$p$" implies "$p$ implies $q$"; that "not-$p$ implies $p$" implies $p$, or that "$p$ implies $q$" implies "$q$ or not-$p$." Each of these assumptions is equivalent to the principle of reduction, and may, if we choose, be substituted for it. Some of them—especially excluded middle and double negation—appear to have far more self-evidence. But when we have seen how to define disjunction and negation in terms of implication, we shall see that the supposed simplicity vanishes, and that, for formal purposes at any rate, reduction is simpler than any of the possible alternatives. For this reason I retain it among my premisses in preference to more usual and more superficially obvious propositions.

**19.** Disjunction or logical addition is defined as follows: "$p$ or $q$" is equivalent to "'$p$ implies $q$' implies $q$." It is easy to persuade ourselves of this equivalence, by remembering that a false proposition implies every other; for if $p$ is false, $p$ does imply $q$, and therefore, if "$p$ implies $q$" implies $q$, it follows that $q$ is true. But this argument again uses principles which have not yet been demonstrated, and is merely designed to elucidate the definition by anticipation. From this definition, by the help of reduction, we can prove that "$p$ or $q$" is equivalent to "$q$ or $p$." An alternative definition, deducible from the above, is: "Any proposition implied by $p$ and implied by $q$ is true," or,

in other words, "'$p$ implies $s$' and '$q$ implies $s$' together imply $s$, whatever $s$ may be." Hence we proceed to the definition of negation: not-$p$ is equivalent to the assertion that $p$ implies all propositions, *i.e.* that "$r$ implies $r$" implies "$p$ implies $r$" whatever $r$ may be*. From this point we can prove the laws of contradiction and excluded middle and double negation, and establish all the formal properties of logical multiplication and addition—the associative, commutative and distributive laws. Thus the logic of propositions is now complete.

Philosophers will object to the above definitions of disjunction and negation on the ground that what we *mean* by these notions is something quite distinct from what the definitions assign as their meanings, and that the equivalences stated in the definitions are, as a matter of fact, significant propositions, not mere indications as to the way in which symbols are going to be used. Such an objection is, I think, well-founded, if the above account is advocated as giving the true philosophic analysis of the matter. But where a purely formal purpose is to be served, any equivalence in which a certain notion appears on one side but not on the other will do for a definition. And the advantage of having before our minds a strictly formal development is that it provides the data for philosophical analysis in a more definite shape than would be otherwise possible. Criticism of the procedure of formal logic, therefore, will be best postponed until the present brief account has been brought to an end.

* The principle that false propositions imply all propositions solves Lewis Carroll's logical paradox in *Mind*, N. S. No. 11 (1894). The assertion made in that paradox is that, if $p$, $q$, $r$ be propositions, and $q$ implies $r$, while $p$ implies that $q$ implies not-$r$, then $p$ must be false, on the supposed ground that "$q$ implies $r$" and "$q$ implies not-$r$" are incompatible. But in virtue of our definition of negation, if $q$ be false both these implications will hold: the two together, in fact, whatever proposition $r$ may be, are equivalent to not-$q$. Thus the only inference warranted by Lewis Carroll's premisses is that if $p$ be true, $q$ must be false, *i.e.* that $p$ implies not-$q$; and this is the conclusion, oddly enough, which common sense would have drawn in the particular case which he discusses.

# CHAPTER III.

## IMPLICATION AND FORMAL IMPLICATION.

**37.** In the preceding chapter I endeavoured to present, briefly and uncritically, all the data, in the shape of formally fundamental ideas and propositions, that pure mathematics requires. In subsequent Parts I shall show that these are all the data by giving definitions of the various mathematical concepts—number, infinity, continuity, the various spaces of geometry, and motion. In the remainder of Part I, I shall give indications, as best I can, of the philosophical problems arising in the analysis of the data, and of the directions in which I imagine these problems to be probably soluble. Some logical notions will be elicited which, though they seem quite fundamental to logic, are not commonly discussed in works on the subject; and thus problems no longer clothed in mathematical symbolism will be presented for the consideration of philosophical logicians.

Two kinds of implication, the material and the formal, were found to be essential to every kind of deduction. In the present chapter I wish to examine and distinguish these two kinds, and to discuss some methods of attempting to analyze the second of them.

In the discussion of inference, it is common to permit the intrusion of a psychological element, and to consider our acquisition of new knowledge by its means. But it is plain that where we validly infer one proposition from another, we do so in virtue of a relation which holds between the two propositions whether we perceive it or not: the mind, in fact, is as purely receptive in inference as common sense supposes it to be in perception of sensible objects. The relation in virtue of which it is possible for us validly to infer is what I call material implication. We have already seen that it would be a vicious circle to define this relation as meaning that *if* one proposition is true, *then* another is true, for *if* and *then* already involve implication. The relation holds, in fact, when it does hold, without any reference to the truth or falsehood of the propositions involved.

But in developing the consequences of our assumptions as to implication, we were led to conclusions which do not by any means agree with

what is commonly held concerning implication, for we found that any false proposition implies every proposition and any true proposition is implied by every proposition. Thus propositions are formally like a set of lengths each of which is one inch or two, and implication is like the relation "equal to or less than" among such lengths. It would certainly not be commonly maintained that "$2+2=4$" can be deduced from "Socrates is a man," or that both are implied by "Socrates is a triangle." But the reluctance to admit such implications is chiefly due, I think, to preoccupation with formal implication, which is a much more familiar notion, and is really before the mind, as a rule, even where material implication is what is explicitly mentioned. In inferences from "Socrates is a man," it is customary not to consider the philosopher who vexed the Athenians, but to regard Socrates merely as a symbol, capable of being replaced by any other man; and only a vulgar prejudice in favour of true propositions stands in the way of replacing Socrates by a number, a table, or a plum-pudding. Nevertheless, wherever, as in Euclid, one particular proposition is deduced from another, material implication is involved, though as a rule the material implication may be regarded as a particular instance of some formal implication, obtained by giving some constant value to the variable or variables involved in the said formal implication. And although, while relations are still regarded with the awe caused by unfamiliarity, it is natural to doubt whether any such relation as implication is to be found, yet, in virtue of the general principles laid down in Section C of the preceding chapter, there must be a relation holding between nothing except propositions, and holding between any two propositions of which either the first is false or the second true. Of the various equivalent relations satisfying these conditions, one is to be called *implication*, and if such a notion seems unfamiliar, that does not suffice to prove that it is illusory.

**38.** At this point, it is necessary to consider a very difficult logical problem, namely, the distinction between a proposition actually asserted, and a proposition considered merely as a complex concept. One of our indemonstrable principles was, it will be remembered, that if the hypothesis in an implication is true, it may be dropped, and the consequent asserted. This principle, it was observed, eludes formal statement, and points to a certain failure of formalism in general. The principle is employed whenever a proposition is said to be *proved*; for what happens is, in all such cases, that the proposition is shown to be implied by some true proposition. Another form in which the principle is constantly employed is the substitution of a constant, satisfying the hypothesis, in the consequent of a formal implication. If $\phi x$ implies $\psi x$ for all values of $x$, and if $a$ is a constant satisfying $\phi x$, we can assert $\psi a$, dropping the true hypothesis $\phi a$. This occurs, for example, whenever any of those rules of inference which employ the hypothesis that the variables involved are propositions, are applied to particular

propositions. The principle in question is, therefore, quite vital to any kind of demonstration.

The independence of this principle is brought out by a consideration of Lewis Carroll's puzzle, "What the Tortoise said to Achilles*." The principles of inference which we accepted lead to the proposition that, if $p$ and $q$ be propositions, then $p$ together with "$p$ implies $q$" implies $q$. At first sight, it might be thought that this would enable us to assert $q$ provided $p$ is true and implies $q$. But the puzzle in question shows that this is not the case, and that, until we have some new principle, we shall only be led into an endless regress of more and more complicated implications, without ever arriving at the assertion of $q$. We need, in fact, the notion of *therefore*, which is quite different from the notion of *implies*, and holds between different entities. In grammar, the distinction is that between a verb and a verbal noun, between, say, "$A$ is greater than $B$" and "$A$'s being greater than $B$." In the first of these, a proposition is actually asserted, whereas in the second it is merely considered. But these are psychological terms, whereas the difference which I desire to express is genuinely logical. It is plain that, if I may be allowed to use the word *assertion* in a non-psychological sense, the proposition "$p$ implies $q$" *asserts* an implication, though it does not *assert* $p$ or $q$. The $p$ and the $q$ which enter into this proposition are not strictly the same as the $p$ or the $q$ which are separate propositions, at least, if they are true. The question is: How does a proposition differ by being actually true from what it would be as an entity if it were not true? It is plain that true and false propositions alike are entities of a kind, but that true propositions have a quality not belonging to false ones, a quality which, in a non-psychological sense, may be called being *asserted*. Yet there are grave difficulties in forming a consistent theory on this point, for if assertion in any way changed a proposition, no proposition which can possibly in any context be unasserted could be true, since when asserted it would become a different proposition. But this is plainly false; for in "$p$ implies $q$," $p$ and $q$ are not asserted, and yet they may be true. Leaving this puzzle to logic, however, we must insist that there is a difference of some kind between an asserted and an unasserted proposition†. When we say *therefore*, we state a relation which can only hold between asserted propositions, and which thus differs from implication. Wherever *therefore* occurs, the hypothesis may be dropped, and the conclusion asserted by itself. This seems to be the first step in answering Lewis Carroll's puzzle.

**39.** It is commonly said that an inference must have premisses and a conclusion, and it is held, apparently, that two or more premisses are necessary, if not to all inferences, yet to most. This view is borne out, at first sight, by obvious facts: every syllogism, for example, is held

* *Mind*, N. S. Vol. IV, p. 278.

† Frege (*loc. cit.*) has a special symbol to denote assertion.

to have two premisses. Now such a theory greatly complicates the relation of implication, since it renders it a relation which may have any number of terms, and is symmetrical with respect to all but one of them, but not symmetrical with respect to that one (the conclusion). This complication is, however, unnecessary, first, because every simultaneous assertion of a number of propositions is itself a single proposition, and secondly, because, by the rule which we called *exportation*, it is always possible to exhibit an implication explicitly as holding between single propositions. To take the first point first: if $k$ be a class of propositions, all the propositions of the class $k$ are asserted by the single proposition "for all values of $x$, if $x$ implies $x$, then '$x$ is a $k$' implies $x$," or, in more ordinary language, "every $k$ is true." And as regards the second point, which assumes the number of premisses to be finite, "$pq$ implies $r$" is equivalent, if $q$ be a proposition, to "$p$ implies that $q$ implies $r$," in which latter form the implications hold explicitly between single propositions. Hence we may safely hold implication to be a relation between two propositions, not a relation of an arbitrary number of premisses to a single conclusion.