MR. LEWIS AND IMPLICATION

THE theory of implication developed by the symbolic logicians seems to have aroused a considerable degree of antagonism among certain students of Logic. There are many philosophers to whom you can not mention the name "Russell," without evoking such comments as, "His logic is purely artificial, for it is nonsense to suppose that a false proposition implies any proposition, or that any proposition implies any true proposition," or, "Who could ever reasonably maintain that, 'The moon is made of green cheese,' implies, 'Caesar died in his bed?' " Most of these critics have not expressed their objections to Mr. Russell's position in black and white, so that it is impossible for us to see just in what the strength and the weakness of their arguments consist; Mr. C. I. Lewis, however, has had the courage of his convictions and has developed his criticisms of the views of Mr. Russell together with certain very interesting logical theories of his own in a series of articles which have appeared partly in *Mind* and partly in this JOURNAL.

The sum and substance of Mr. Lewis's objections to Mr. Russell is this: Mr. Russell, following the older symbolic logicians, holds that a false proposition implies any proposition and that any proposition implies any true proposition. That is, p implies q if either p is false or q is true. Mr. Lewis claims, and not without a certain degree of justice, that this is not what we ordinarily mean by implication. We do not, for example, usually say that, "Socrates was a solar myth," implies, "All triangles have two or more sides." Therefore, as Mr. Lewis tells us, "Not only does the calculus of implication contain false theorems, but all its theorems are not proved. For the theorems are implied by the postulates in the sense of 'implies' which the system uses. Hence *it has not been demonstrated* that the theorems can be inferred from the postulates, even if all the postulates are granted. The assumptions, *e. g.*, of the 'Principia Mathematica' imply the theorems in the same sense that a false proposition implies anything."[1]

Mr. Lewis's reasoning here is fallacious, and the fallacy he commits is that of denying the antecedent. From the fact that if a set of postulates deals correctly with our ordinary relation of inference, it will yield us a correct logic, he infers that if a set of postulates fails to deal with this relation, and, like the Russellian logic, seizes upon some other relation as its fundamental notion, the logic to which it leads must be faulty and incorrect. This is a manifest and a grave error; it is conceivable that we may develop a valid theory

[1] This JOURNAL, Vol. X., p. 242.

of demonstration, the fundamental notion of which is other than what we ordinarily call inference, which is correctly derived from its own postulates. It is not necessary that a theory whose purpose it is to yield us a norm of valid inference should itself in the first instance be a theory of inference. We say that one proposition can be inferred from another if there is a certain relation between them such that we are compelled to accept the former proposition as true if we accept the latter one. The purpose of Logic, in so far as Logic is a norm of inference, is to provide us with certain methods which, when applied to any true proposition of a suitable sort, will yield us other true propositions. These methods need not of themselves involve any reference to the concept of inference, and may not lead us to realize that they are methods of inference. Indeed, they can not lead us to realize that they are methods of inference, even if they actually concern themselves with such methods, for then they would form a portion of their own subject-matter, and we should be involved in that philosphical lifting of oneself up by one's boot-straps, the pernicious consequences of which have been pointed out so well by Mr. Bertrand Russell in that part of the "Principia Mathematica" which deals with the theory of types. If the natural history of the process of inference is a branch of Logic, it is a Logic of a very different type from that which it is the purpose of the logisticians to develop, and there is no reason under the sun why this latter Logic should be doomed under penalty of death to make use of our every-day notion of implication.

The only questions, then, which can reasonably be asked concerning the correctness of the Russellian Logic are, Is it actually a correct norm of valid inference? and, Is the coherence and self-consistency which it claims for itself, and tries to justify by an orderly derivation of its theorems from a small and simple set of postulates, genuine or factitious? As we have already seen, Mr. Lewis answers both of these questions in a manner adverse to the claims of Mr. Russell, but we have found his arguments to be fallacious. To arrive at the true answer to these questions, we must discuss what the function of postulates is in a deductive system such as the Russellian logic. Now, the main function of the postulates of any system is to stand as hostages for the system; they must be statements, the acceptance of which commits one to the acceptance of the entire system as true. That is, we must be able to affirm that the system of propositions is true, unless, by chance, the postulates should fail to be satisfied. This is not to be taken, as Mr. Lewis seems to be in danger of taking it, as the assertion of any occult connection or motive force acting between the postulates and the system: there is no need for us to suppose that the truth of the propositions is conditioned in any causal way by the truth of the postulates. That either the postulates are false, or the system of propositions is true[2]—this is all we need know concerning the relation which the postulates of Logic bear to the system of logical truths, since this is enough to secure for us that the truth of the propositions of Logic must be maintained by any one who believes in its postulates. It matters not whether the disjunction expressed in this last proposition be what Mr. Lewis calls "extensional" or what he calls "intensional":—if it is a mere extensional disjunction, such as can hold between two unrelated and mutually irrelevant propositions, then we may have attained our knowledge of the truth of the propositions of logic independently of our postulates, but we shall have attained it, nevertheless. So long as the conditions of the truth of our theorems are not incorrectly extended and expanded, the question of the genesis of such propositions as we accept as true has no interest for us, except in so far as it is bound up with the question of their validity. The knowledge of the postulates must be a sufficient ground for a knowledge of the theorems:—if we find that it is more than sufficient, and that we do not need a previous knowledge of the postulates to attain to that of the theorems, so much the better. If, that is, we interpret Mr. Lewis as maintaining that we are justified in inferring one proposition from another whenever we are able to proceed to the first from the second by a valid process of reasoning, then, since we are clearly bound to accept the Russellian theorems in the Algebra of Logic if we accept the Russellian postulates, we must maintain that the Russellian postulates

[2] In an article entitled, "A Too Brief Set of Postulates for the Algebra of Logic" (This JOURNAL, Volume XII., p. 523), Mr. Lewis makes what practically amounts to the claim that the postulate, "Any true proposition implies all true propositions," is a sufficient basis for the whole of logic, or rather, that the methods of the Russellians should lead them to this conclusion, which Mr. Lewis regards as very objectionable. The grounds on which Mr. Lewis bases this claim are that since this postulate is true, and since it tells us that any true proposition implies any true proposition, it implies (in its own sense, which is also that of Mr. Russell) any true proposition whatever. There is no question that Mr. Lewis's postulate does actually imply any true proposition, but this is not the entire function which a postulate must fill. The fact that any proposition in a mathematical system, so it seems, can be made a member of some set of postulates or other, simply goes to show that the primacy of the postulates of a mathematical system is a primacy in the order of knowledge, not in the order of existence. I need not say, however, that even this primacy of the postulates of a system refers only to their status within a given investigation. Now, it is obvious that this priority in the order of knowledge can not be claimed for Mr. Lewis's postulate. It is for this reason that I have said that the fact that either a theorem must be true or a set of postulates false is all that we must *know* for us to be able to say that they imply the theorem, and not that it is all that need be true to render the latter statement true in its usual sense.

imply the theorems, not only according to their own peculiar definition of the relation of implication (which may, indeed, have but little in common with the ordinary definition of that relation), but precisely according to our usual understanding of the relation of implication. Now, this is taken by Mr. Lewis himself as the ultimate criterion of the true nature of implication.

It may be objected by some that the relation of implication by which we obtain the theorems of logic from its postulates—or, indeed, any theorems from any postulates—is not merely the Russellian relation of material implication, which a false proposition bears to any proposition and any proposition to a true one. As Mr. Lewis says, "Euclid's parallel postulate or Lobachevski's postulate about coplanars is—one of them—false. Nevertheless, he errs who would take either postulate to imply anything and everything. Logical consequences follow regardless of truth or falsity of premises."[3] (Mr. Lewis's statement here is unquestionably true. Mr. Russell's material implication is not adequate to the deduction of the theorems of geometry from their premises, and it might be thought that it is still less adequate to the deduction of the theorems and propositions of logic from Mr. Russell's set of primitive propositions. This, however, is not the case. There is, as Mr. Lewis himself recognizes, a vast difference between the postulates of logic and the postulates of geometry. "Pure mathematics is not concerned with the truth either of postulates or of theorems: so much is an old story . . . . Indeed, the attempt to separate formal consistency and material truth is, in the case of the logic, peculiarly difficult."[4]) That is, pure mathematics is concerned only with the question whether the theorems follow from the postulates, while logic must take its postulates and theorems as both true. A mathematical set of postulates can be investigated without any reference to an actual system which in reality embodies them, while the postulates of logic, in so far as they remain postulates of logic, must be regarded as actually embodied in the constitution of every system. The postulates of geometry are hypotheses, or *types of possible truths;* the postulates of logic, if they are correctly stated, are *truths.* Now, a hypothesis or type of truths is not a proposition. The postulates of geometry, *qua* postulates of geometry, may apply indifferently to points in space or to number-triads or to any other sort of entity you please, and are on this very account neither true nor false in themselves, but only in their particular manifestations. Since they are neither true nor false, and hence not propositions, it is nonsense to speak of their implying anything or being implied by anything in the same way in which we can speak of propositions,

[3] This JOURNAL, Vol. X., p. 432.

[4] *Ibid.*, p. 429.

such as the postulates of logic, as implying or being implied by something. It is only natural, then, that it should be permissible to use methods in deducing the theorems of logic from their postulates which are prohibited in the case of the postulates and theorems of geometry. It is by virtue of the very distinction between logic and other branches of mathematics, which Mr. Lewis stresses so strongly, that the objections which he raises against Mr. Russell's employment of the relation of material implication in the development of his theorems may be shown to be irrelevant.

The relation of implication which the postulates of geometry bear to its theorems deserves a further consideration, for a misapprehension of the nature of this relation is at the bottom of most of Mr. Lewis's errors. The postulates of geometry, as we have seen, are not propositions, but blank forms of propositions, which may be filled in in a countless number of different ways. Thus the postulate, "Any two points are connected by one and only one line," may be filled in with a content consisting of actual spatial entities, or of number-triads (in place of points) and pairs of linear equations in three unknown quantities (in place of lines), or with any one of an infinity of other possible specific determinations. Now, Mr. Russell denotes a blank form for propositions by the names, "universal" and "propositional function." Moreover, he has given a definite and complete discussion of the relation of implication between propositional functions, which he calls the relation of *formal implication*. A propositional function ϕ is said to imply another ψ formally if every entity which fills out the blank form ϕ into a true proposition also fills out the blank form ψ into a true proposition.[5] Thus to say that the postulates of geometry imply the theorems formally is to say that every system which satisfies the postulates of geometry also satisfies the theorems of geometry. This is manifestly true of the postulates and theorems of our geometry, and constitutes a necessary condition for the validity of the latter. A little reflection will convince us that it is also a *sufficient* condition, for otherwise we should have theorems not implied by the postulates of geometry, but true of all geometrical systems. This means of all *possible* geometrical systems, for a geometrical system is a universal, and the possibility of a universal is identical with its actuality. Now, we should ordinarily say that any proposition follows from a given set of postulates, if it is true of every possible system which satisfies these postulates, on the ground of the very nature of universals themselves. As a consequence, we see that Mr. Russell's notion of formal implication among universals is in every respect in harmony with our every-day use of the term.

[5] Whitehead and Russell, "Principia Mathematica," p. 21.

It is this relation of formal implication which Mr. Russell always uses when he derives the theorems of any mathematical system, such as that of linear order,[6] from their postulates, and the correctness of his view of the nature of implication is substantiated by the fact that in every case his results have agreed with those of other mathematicians who have made use of the same postulates.

It may be objected to this analysis of the relation of implication between postulate and theorem, that the relation of implication still holds, and differs from that of material implication, when our postulates are propositions concerning specific objects, and not mere propositional functions which may apply to anything at all. I think, however, that it is only in so far as they are taken as the representatives of propositional functions that propositions imply anything in this sense. When I deduce the properties of actual space from Euclid's postulates, I am really deducing conclusions from certain laws which an infinity of systems may satisfy, and which our space does satisfy, and I am applying them to our space. The possibility of a reference to other systems plays an essential part in the deduction. So, too, when I say, "If it rains, I shall get wet," the real implication which I desire to assert is, "*In any situation such as the present*, when it is raining on me, I get wet." The tacit, "other things being equal," which may always be prefixed to such implications, points out the universality of reference which such implications are intended to have, as will be seen if we write it, "whenever other things are equal." This explains the unnaturalness of such material implications as, "If two and two are four, Caesar is still alive," for there is no obvious general law or formal implication to which they may be reduced as instances.

The reason why Mr. Lewis seems unable to understand the significance or even the nature of Mr. Russell's formal implication appears to be that he ignores the distinction between propositions and propositional functions. Hence he regards $(x): \phi x \cdot \mathsf{U} \cdot \psi x$, the relation of formal implication between two laws, as but a particular instance of what he calls "strict implication," which relates propositions.[7] Further on in the same article, he speaks of "cases" of the truth or falsity of a proposition, while a proposition, unlike a propositional function, is either simply true or simply false, and can have no instances of truth or falsity. These slips are particularly regrettable in the case of a man who has published so extensively on logistical matters as Mr. Lewis has. If Mr. Lewis disagrees with Mr. Russell's distinction between propositions and propositional functions, he should have made the fact of this disagreement clear before setting forth on his

[6] "Principia Mathematica," Vol. II., Part V.

[7] This JOURNAL, Vol. X., p. 430, note.

criticism of Mr. Russell's logistical views, as this distinction is the heart and soul of the Russellian logic.

Mr. Lewis's arguments against Mr. Russell have little logical cogency, and one feels that they were developed to give an excuse for his constructive work in the definition of his system of "strict implication," which really requires no such apology for its existence. It is not the place here for me to comment upon this exceedingly valuable and interesting piece of work, except to remark that its logical worth is utterly independent of an acceptance of Mr. Lewis's contentions against Mr. Russell. One may grant that the Russellian system is both a logic and a self-subsistent logic, and yet realize the obvious fact, which it is to Mr. Lewis's credit to have noticed, that it is unable to distinguish between the notion of truth, pure and simple, and the notion of that truth which results as a consequence of the laws of logic alone.[8] The logic developed by Mr. Lewis is able to give an account of this notion, and is, in so far, more complete in its apparatus—though not necessarily more *correct* in any way—than that of Mr. Russell. In the form which it finally assumes, it starts with a distinction between *de facto* and necessary falsity, from which a natural transition leads us to the notions of necessary and *de facto* truth, disjunction, and implication. The whole theory is carried out with great patience and ingenuity, and taking it all in all, with logical correctness.[9] It is, however, a supplement to the Russellian logic, and not a refutation of it.

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SOCIETIES

NEW YORK BRANCH OF THE AMERICAN PSYCHOLOGICAL ASSOCIATION

THE New York Branch of the American Psychological Association met in conjunction with the Section of Anthropology and Psychology of the New York Academy of Sciences, on Monday, May 1, at Columbia University. The following are abstracts of the papers read:

[8] Cf. "The Matrix Algebra for Implications," this JOURNAL, Vol. XI., pp. 589–600. This represents the most finished form of Mr. Lewis's system.

[9] Mr. Lewis fails to include among his postulates any one, such as $\sim(\sim p = -p)$, which would definitely distinguish his algebra from the traditional logical calculus. He also states it as a self-evident principle that mutually eqivalent propositions may be substituted for one another, whereas this is capable of proof, so far as the operations of his algebra are themselves concerned, on the basis of his postulates, and stands in need of such an explicit proof. These are, however, only minor inadequacies, and are in no way faults of the Lewis "Algebra of Logic" itself.